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VARIATIONS ON A THEME OF SOLOMON

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CONTENTS.

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ABSTRACT.

The main aim of this thesis is to examine the structure of the Hecke algebra $H_K(G,B)$ of a finite group G with a split (B,N) pair of rank n and characteristic p, with Weyl group W, over a field K of characteristic p. We then see how this relates to the structure of the KG-module L induced from the principal KB-module.

Chapters 1, 2 and 3 are introductory. Chapter 1 deals with the properties of finite Coxeter groups, and gives L. Solomon's decomposition of the group algebra of a finite Coxeter group over the field of rational numbers. In Chapter 2 we discuss finite groups with split (B,N) pairs and their irreducible modular representations. Chapter 3 deals with Hecke algebras and the generic ring and some of its specialisations.

In Chapter 4, we examine the structure of the 0-Hecke algebra H of type (W,R) over any field, which is defined in Chapter 3; the algebra $H_K(G,B)$ is an example of one of these. H has 2^n distinct irreducible representations where $n = |R|$, all of which are one-dimensional, and correspond in a natural way with subsets of R. H can be written as a direct sum of 2^n indecomposable left (or right) ideals, in a similar manner to the Solomon decomposition of the underlying Weyl group W.

In Chapter 5, we also obtain decompositions of the generic ring similar to Solomon's decomposition of the underlying Coxeter group. These decompositions carry over to some specialisations of the generic ring; in particular, we get Solomon's decomposition of the Coxeter group and decompositions of the Hecke algebra of a finite group with (B,N) pair over a field of characteristic zero.

Certain homology modules which arise from the Tits building of a finite group with a (B,N) pair, called relative Steinberg modules because of the character of the group they afford, are discussed in Chapter 6. Finally, in Chapter 7, we see that L is a direct sum of 2^n indecomposable KG-modules, each of which is a relative Steinberg module. We can deduce what $|W|$ of the composition factors of L are, but there are in general others.

NOTATION.

$\mathbb{Z}$	The rational integers.
$\mathbb{Z}^+$	The non-negative rational integers.
$\mathbb{Q}$	The rational number field.
$\mathbb{R}$	The real number field.
DCC	Descending chain condition.
ACC	Ascending chain condition.
K^*	If K is a field, $K^* = K - \{0\}$.

Let S be a set and J a subset of S. Then:

$\lvert J \rvert$	The number of elements in J.
$\hat{J}$	The complement of J in S.
$\{i_1,..\hat{i}_s,..i_n\}$	The set $\{i_1,..i_n\} - \{i_s\}$.
$< J >$	The group or the algebra generated by J.

$w_1..\hat{w}_j..w_n = w_1..w_{j-1}w_{j+1}..w_n$

$\cup$	Set union.
$\cap$	Set intersection.

$(w_iw_jw_i..)_n$ and $(w_iw_j..)_n$ The product of the first n terms of the sequence $w_i,w_j,w_i,w_j,..$

$(..w_iw_jw_i)_n$ and $(..w_jw_i)_n$ The product of the first n terms of the alternating sequence $w_i,w_j,w_i,w_j,..$ written from right to left.

Chapter 1: FINITE COXETER GROUPS.

(1.1) Finite Reflection Groups.

Most of the results in this section are proved in either Bourbaki [1], Carter [3] or Steinberg [24].

Let V be an n-dimensional real Euclidean space, with a positive definite symmetric scalar product (,). An orthogonal linear transformation w of V is called a reflection if $w \neq 1$ and w fixes some hyperplane H pointwise. If r is a vector perpendicular to H, then w is given by

(1.1.1) $$w(x) = x - \frac{2(x,r)}{(r,r)}r \quad \text{for all } x \in V.$$

We write $w=w_r$. Note that $w_r^2=1$ and $w_r(r)=-r$. w_r is an automorphism of V, and if σ is an automorphism of V, then

$$\sigma w_r \sigma^{-1} = w_{\sigma(r)}.$$

We call a group W a reflection group if W is generated by a set of reflections, and $|W| < \infty$.

(1.1.2) <u>Definition</u>: A finite subset ϕ of V is called a system of roots if the following conditions are satisfied:

(1) ϕ is a set of generators for V.

(2) $0 \notin \phi$, and if $r \in \phi$ and $cr \in \phi$ for some $c \in \mathbb{R}$, then $c = \pm 1$.

(3) $w_r(\phi) = \phi$ for each $r \in \phi$, where w_r is the reflection defined in (1.1.1).

The elements of ϕ are called roots, and from the definition it follows that $\phi = -\phi$. Let $W = W(\phi)$ be the group generated by the reflections $\{w_r : r \in \phi\}$. Then the restriction of W to ϕ is faithful, and hence W is a finite group. W is called the Weyl group of the root system ϕ.

NOTE: Bourbaki [1] and Carter [3] have the following additional condition to define a system of roots:

(4) $\frac{2(r,s)}{(r,r)} \in Z$ for all r and $s \in \phi$.

We do not include this condition in our definition as we wish to associate root systems with arbitrary finite reflection groups.

(1.1.3) LEMMA (Curtis [9], Richen [18]): Every finite group W generated by reflections of a real Euclidean space V can be identified with the Weyl group of some root system.

Proof: First, by dividing out the space of invariant vectors under W, we may assume that W is a finite group generated by reflections, which leaves no non-zero vectors fixed. Let ϕ be the set of all unit vectors which are perpendicular to the hyperplane fixed by some reflection in W. Then

(1) ϕ is a system of roots. For if $r,s \in \phi$, then $w_r(s) \in \phi$ as $w_r(s)$ is a unit vector orthogonal to the hyperplane fixed pointwise by the reflection $w_r w_s w_r^{-1} = w_{w_r(s)}$. So ϕ is finite and generates V, and r and $cr \in \phi$ implies $c=\pm 1$, and $r \in \phi$ implies $w_r \in W$.

(2) The w_r's for $r \in \phi$ generate W.

(1.1.4) Definition: A subset ϕ^+ of roots of ϕ is called a positive system if it consists of the roots which are positive relative to some ordering of V.

(1.1.5) Definition: A subset $\prod$ of roots of ϕ is called a simple system if

(1) the elements of $\prod$ are linearly independent, and

(2) every root is a linear combination of the elements of $\prod$ in which the coefficients are either all non-negative or all non-positive.

(1.1.6) PROPOSITION: Each simple system is contained in a unique positive system, and each positive system contains a unique simple system.

Fix a simple system $\prod$ of ϕ. Then $\prod$ determines a partial order relation $>$ on V, for which the non-negative elements are $\sum_{a \in \prod} c_a a$, with all $c_a \geq 0$. Let

$$\phi^+ = \{a \in \phi\colon a > 0\}, \qquad \phi^- = \{a \in \phi\colon a < 0\},$$

the positive and negative roots respectively.

(1.1.7) LEMMA: Let $a \in \prod$. Then $w_a(a) = -a$ and $w_a(\phi^+ - \{a\}) = \phi^+ - \{a\}$.

(1.1.8) LEMMA: Let W_o be the subgroup of $W(\phi)$ generated by the reflections $\{w_a\colon a \in \prod\}$. Then $W_o(\prod) = \phi$ and $W_o = W(\phi)$.

(1.1.9) Definition: Let $\prod$ be a set of simple roots of ϕ.

The corresponding set of reflections $R = \{w_a: a \in \Pi\}$ is called the set of fundamental reflections, and w_a $(a \in \Pi)$ is called a fundamental reflection. We write $\Pi = \{r_1, r_2, .. r_n\}$ and $R = \{w_1, w_2, .. w_n\}$, where $w_i = w_{r_i}$ and $n = \dim V$.

(1.1.10) Definition: For $w \in W$, let $l(w)$ be the minimal length of all possible expressions of w as a product of the fundamental reflections: we say $l(w)$ is the 'length of w'. An expression $w = w_{i_1} w_{i_2} .. w_{i_t}$, with $w_{i_j} \in R$ for all j, $1 \leqslant j \leqslant t$, is called reduced if $l(w) = t$.

(1.1.11) Definition: For $w \in W$, define

$$n(w) = |\{a \in \phi^+ : w(a) \in \phi^-\}|.$$

Let $\phi_w^+ = \{a \in \phi^+ : w(a) \in \phi^+\}$, $\phi_w^- = \{a \in \phi^+ : w(a) \in \phi^-\}$.

(1.1.12) PROPOSITION:

(1) For all $w \in W$, $l(w) = l(w^{-1})$, $n(w) = n(w^{-1})$, and $l(w) = n(w)$.

(2) Let $w \in W$, and $w \neq 1$. Then $|\phi_w^-| \geqslant 1$.

(3) For all $w \in W$, $w_i \in R$,

$$l(w_i w) = l(w) + 1 \quad \text{if} \quad w^{-1}(r_i) \in \phi^+$$
$$l(w_i w) = l(w) - 1 \quad \text{if} \quad w^{-1}(r_i) \in \phi^-$$

$$l(w w_i) = l(w) + 1 \quad \text{if} \quad w(r_i) \in \phi^+$$
$$l(w w_i) = l(w) - 1 \quad \text{if} \quad w(r_i) \in \phi^-$$

(4) Let $w = w_{i_1} .. w_{i_t}$, with $w_{i_j} \in R$ for all j, $1 \leqslant j \leqslant t$. If $l(w) < t$, then for some j and k, $1 \leqslant j < k \leqslant t-1$, we have

(a) $r_{i_j} = w_{i_{j+1}} .. w_{i_k}(r_{i_{k+1}})$

(b) $w_{i_{j+1}} .. w_{i_{k+1}} = w_{i_j} .. w_{i_k}$.

(c) $w = w_{i_1} .. \hat{w}_{i_j} .. \hat{w}_{i_k} .. w_{i_t}$.

(5) For $w \in W$, $r_i \in \prod$,

(a) $w(r_i) \in \phi^-$ if and only if there is a reduced expression for w ending with w_i.

(b) $w^{-1}(r_i) \in \phi^-$ if and only if there is a reduced expression for w beginning with w_i.

(6) Suppose $w_{i_1}..w_{i_t}$ is a reduced expression but $w_{i_0}w_{i_1}..w_{i_t}$ is not reduced (where $w_{i_j} \in R$ for all j, $0 \leq j \leq t$). Then for some k, $1 \leq k \leq t$, $w_{i_1}..w_{i_k} = w_{i_0}w_{i_1}..w_{i_{k-1}}$.

(7) We can pass between any two reduced expressions for $w \in W$ by substitutions of the form

$$(w_i w_j w_i ..)_{n_{ij}} = (w_j w_i w_j ..)_{n_{ij}}$$

where w_i, $w_j \in R$, $w_i \neq w_j$, and n_{ij}= order of $w_i w_j$ in W.

(8) For $w \in W$, if $w(\phi^+) = \phi^+$ or $w(\prod) = \prod$ or $n(w) = 0$, then $w = 1$.

(9) There is a unique element $w_0 \in W$ of maximal length, with the properties that $w_0^2 = 1$, $w_0(\phi^+) = \phi^-$, $w_0(\prod) = -\prod$ and for all $w_i \in R$, there exists $w_j \in R$ such that $w_0 w_i w_0 = w_j$.

(10) For any $v \in V$, $w \in W$, $w(v) = v$ if and only if w is a product of reflections corresponding to roots perpendicular to v.

(1.2) Finite Coxeter Systems.

The results in this section are proved in either Bourbaki [1], Carter [3] or Steinberg [24].

(1.2.1) Definition: A Coxeter system (W,R) is a group W generated by a finite set of involutions R, such that W has a presentation

$$W = \langle w_i \in R: (w_i w_j)^{n_{ij}} = 1 \text{ with } n_{ii} = 1,\ n_{ij} \geq 2 \text{ if } i \neq j \rangle$$

(W,R) is a finite Coxeter system if $|W| < \infty$. In this case W is called a finite Coxeter group.

(1.2.2) THEOREM: Let W be a finite group generated by a set of involutions $R = \{w_i : 1 \leq i \leq n\}$ satisfying the following 'exchange condition':

(E) If $w_{i_1} \ldots w_{i_k}$ is a reduced expression but $w_{i_0} w_{i_1} \ldots w_{i_k}$ is not, where $w_{i_j} \in R$ for $0 \leq j \leq k$, then for some m, $1 \leq m \leq k$, $w_{i_1} \ldots w_{i_m} = w_{i_0} w_{i_1} \ldots w_{i_{m-1}}$.

(a) Then (W,R) is a Coxeter system, i.e. W has a presentation as an abstract group

$$W = \langle w_i \in R: (w_i w_j)^{n_{ij}} = 1, \text{ with } n_{ii} = 1,\ n_{ij} \geq 2 \text{ if } i \neq j \rangle.$$

(b) Let $W(\Phi)$ be the Weyl group of a system of roots, with R the set of fundamental reflections. Then $(W(\Phi),R)$ is a Coxeter system.

(c) To every Coxeter system (W,R), where W is a finite group, there corresponds a root system Φ, and there is an isomorphism $T: W \to W(\Phi)$ such that $T(R)$ is the set of fundamental reflections of $W(\Phi)$ for some set of simple roots

Π of ϕ.

Hence we can identify a Coxeter group W with a Euclidean reflection group, and thus speak of the corresponding root system ϕ and simple system Π.

(1.2.3) <u>THEOREM</u>: w_i, $w_j \in R$ are conjugate in the finite Coxeter group W if and only if there exists a sequence $w_{k_1}=w_i$, $w_{k_2}, \ldots w_{k_s}=w_j$ with $w_{k_t} \in R$ for all t, $1 \leq t \leq s$, such that $w_{k_t} w_{k_{t+1}}$ has odd order for all t, $1 \leq t \leq s-1$.

(1.2.4) <u>THEOREM</u>: Let (W,R) be a Coxeter system. For $J \subseteq R$, let $W_J = < w_j \in J >$, and let Π_J be the set of simple roots corresponding to J. Let V_J be the subspace of V spanned by the elements of Π_J, and let $\phi_J = \phi \cap V_J$. Then:

(1) ϕ_J is a root system.

(2) Π_J is a simple system in ϕ_J.

(3) W_J is the Weyl group of ϕ_J.

(4) (W_J, J) is a Coxeter system.

(5) If l_J denotes the length function on W_J, then for all $w \in W_J$, $l_J(w) = l(w)$.

(1.2.5) <u>THEOREM</u>: $J \to W_J$ is a lattice isomorphism from the collection of subsets of Π $(\cup,\cap)$ to the collection of subgroups of W $(< >, \cap)$, i.e. for all $J,K \subseteq R$,

$$W_{J \cup K} = < W_J, W_K >$$

$$W_{J \cap K} = W_J \cap W_K.$$

(1.2.6) <u>Definition</u>: (W,R) is decomposable if there exists a subset J of R such that for all $w_j \in J$ and for all $w_k \in K$, where $K = R-J$, $w_j w_k = w_k w_j$. In this case, $W = W_J \times W_K$ (a direct product of groups). Otherwise, (W,R) is indecomposable.

The finite Coxeter systems have been classified - see Bourbaki [1]. In Appendix 1, we give the classification of the finite indecomposable Coxeter systems.

(1.3) <u>Distinguished Coset Representatives</u>.

Proofs of results in this section can be found in Solomon [20], if not given below.

Let (W,R) be a finite Coxeter system.

(1.3.1) <u>Definition</u>: For all $J \subseteq R$, define the sets

$$X_J = \{w \in W : w(\Pi_J) \subseteq \Phi^+\}$$

$$Y_J = \{w \in W : w(\Pi_J) \subseteq \Phi^+,\ w(\Pi_{\hat{J}}) \subseteq \Phi^-\}$$

where $\hat{J} = R-J$.

Note that $X_J = \bigcup_{J \subseteq K \subseteq R} Y_K$, and for distinct subsets J_1, J_2 of R, $Y_{J_1} \cap Y_{J_2} = \emptyset$, the empty set.

(1.3.2) <u>LEMMA</u>: The set X_J is a set of left coset representatives for W mod W_J. If $w \in W$ and $w = xu$ with $x \in X_J$, $u \in W_J$, then $l(w) = l(x)+l(u)$.

Let w_{oJ} be the unique element of maximal length in W_J, for all $J \subseteq R$.

(1.3.3) <u>LEMMA</u>: Let $J \subseteq R$ and let $\hat{J}$ be its complement in R. If $y \in Y_J$, then $yw_{o\hat{J}} \in X_{\hat{J}}$ and $l(y) = l(yw_{o\hat{J}})+l(w_{o\hat{J}})$.

Proof: $w_{o\hat{J}}(\Pi_{\hat{J}}) = -\Pi_{\hat{J}}$. Since $y \in Y_J$, $y(\Pi_{\hat{J}}) \subset \Phi^-$ and hence $yw_{o\hat{J}}(\Pi_{\hat{J}}) \subset \Phi^+$. So $yw_{o\hat{J}} \in X_{\hat{J}}$.

Then $y = (yw_{o\hat{J}})w_{o\hat{J}}$ with $yw_{o\hat{J}} \in X_{\hat{J}}$, $w_{o\hat{J}} \in W_{\hat{J}}$. By lemma (1.3.2), $l(y) = l(yw_{o\hat{J}}) + l(w_{o\hat{J}})$.

(1.3.4) COROLLARY: If $y \in Y_J$, then $y = ww_{o\hat{J}}$ with $l(y) = l(w) + l(w_{o\hat{J}})$ and $w \in X_{\hat{J}}$.

(1.3.5) LEMMA: $w_{o\hat{J}}$ is the unique element of minimal length in Y_J, and $w_o w_{oJ}$ is the unique element of maximal length in Y_J.

Proof: By corollary (1.3.4), $w_{o\hat{J}}$ is the unique element of minimal length in Y_J. Consider the map $f:Y_J \to Y_{\hat{J}}$ given by $f(y) = w_o y$ and the map $g:Y_{\hat{J}} \to Y_J$ given by $g(x) = w_o x$. Then f and g are mutually inverse isomorphisms of the sets Y_J and $Y_{\hat{J}}$, so $|Y_J| = |Y_{\hat{J}}|$. Since $Y_{\hat{J}}$ has w_{oJ} as its unique element of minimal length, it follows that $w_o w_{oJ}$ is the unique element of maximal length in Y_J.

(1.3.6) COROLLARY: $|Y_J| = |Y_{\hat{J}}|$ for all $J \subsetneq R$.

(1.3.7) PROPOSITION: (1) Let $w \in W$ satisfy

(a) $w(\Pi_{\hat{J}}) \subsetneq \Phi^-$

(b) $w(r_j) = r_k$ for all $r_j \in \Pi_J$, and some $r_k \in \Pi$.

Then w is the unique element of maximal length in Y_J, and conversely, the unique element of maximal length in Y_J satisfies (a) and (b).

(2) Let $w \in W$ satisfy

(a) $w(\Pi_J) \subseteq \phi^+$

(b) $w(r_j) = -r_k$ for all $r_j \in \Pi_{\hat{J}}$, and some $r_k \in \Pi$

Then w is the unique element of minimal length in Y_J, and conversely, the unique element of minimal length in Y_J satisfies (a) and (b).

Proof: We have that $w_o w_{oJ}$ is the unique element of maximal length in Y_J, and satisfies (a) and (b) of (1), and that $w_{o\hat{J}}$ is the unique element of minimal length in Y_J, and satisfies (a) and (b) of (2).

(1) We prove that $w = w_o w_{oJ}$. Clearly $ww_{oJ}(\Pi_J) = -w(\Pi_J) \subseteq \phi^-$. Let $r_i \in \Pi_{\hat{J}}$. Then $ww_{oJ}(r_i) = w(r_i + s)$, where s is a linear combination of the elements of Π_J, with all coefficients non-negative. Let $\Pi_J = \{r_{j_1}, ..r_{j_t}\}$. Let $r_{j_k}*$ be the element of Π such that $w(r_{j_k}) = r_{j_k}*$. Let $\Pi_{J*} = \{r_{j_k}*: 1 \leq k \leq t\}$. Then $w(s) \in V_{J*}$. So for $ww_{oJ}(r_i) \in \phi^+$, we must have that $w(r_i) \in V_{J*}$. But each element of V_{J*} is the image under w of an element of V_J. As w is an automorphism of V, $w(r_i) \notin V_{J*}$, and so $ww_{oJ}(\Pi) \subseteq \phi^-$; hence $ww_{oJ} = w_o$.

(2) Show $ww_{o\hat{J}} = 1$, using an argument similar to the above.

In Appendix 2, we give some examples of Coxeter groups, and calculate the subsets Y_J for all $J \subseteq R$.

(1.3.8) PROPOSITION: Let $w \in W$, with $w = xu$, where $x \in X_J$, $u \in W_J$, and $l(w) = l(x) + l(u)$. Suppose $w(r_i) \in \phi^+$. Then

(1) if $w_i \in W_J$, $ww_i = xv$ where $v = uw_i \in W_J$ and $l(ww_i) = l(w) + 1 = l(x) + l(v)$.

(2) if $w_i \notin W_J$ then $ww_i = yv$ where $y \in X_J$, $v \in W_J$, $l(ww_i) = l(y) + l(v)$ and $l(y) > l(x)$.

Proof: (1) Clear since $w(r_i) \in \phi^+$ and $w_i \in W_J$.

(2) $ww_i = yv = xuw_i$. Now, $l(ww_i) = l(w) + 1 = l(xu) + 1 = l(x) + l(u) + 1$. Since $yv(r_i) \in \phi^-$, let $y = w_{i_1}..w_{i_r}$, where $l(y) = r$, and $v = w_{i_{r+1}}..w_{i_s}$, where $l(v) = s-r$, and with $w_{i_j} \in R$ for all j, $1 \leqslant j \leqslant s$. Then $yv = w_{i_1}..w_{i_s}$ is reduced but yvw_i is not, and so there exists j, $1 \leqslant j \leqslant s$ such that

$$w_{i_j}..w_{i_s} = w_{i_{j+1}}..w_{i_s}w_i.$$

There are now two cases to consider.

(a) $r+1 \leqslant j \leqslant s$. Then $yv = yw_{i_{r+1}}..w_{i_j}..w_{i_s}w_i$

That is, $w_{i_j}..w_{i_s} = w_{i_{j+1}}..w_{i_s}w_i$. Then $w_{i_j}..w_{i_s}(r_i) \in \phi^-$. But $w_{i_j}..w_{i_s} \in W_J$, and $r_i \notin \prod_J$ – contradiction. So we must have:

(b) $1 \leqslant j \leqslant r$. Then $yv = w_{i_1}..\hat{w}_{i_j}..w_{i_r}vw_i = xuw_i$.

Thus $w_{i_1}..\hat{w}_{i_j}..w_{i_r}v = xu$. Since v, $u \in W_J$, then by the uniqueness of the expression of w as a product of elements xu, with $x \in X_J$, $u \in W_J$, we have $l(v) \leqslant l(u)$, and thus $l(x) < l(y)$.

(1.3.9) LEMMA: If $K \subsetneq R$, then $|Y_K| > 1$ unless K and $\hat{K}$ are mutually commuting sets (i.e. unless for all $w_i \in K$ and all $w_j \in \hat{K}$ we have that $w_iw_j = w_jw_i$). If K and $\hat{K}$ are mutually commuting sets, then $|Y_K| = |Y_{\hat{K}}| = 1$. In particular, $\emptyset$ and R are mutually commuting sets.

Proof: Obviously $|Y_K| \geqslant 1$, as $w_{o\hat{K}} \in Y_K$.

(a) Suppose there exists $w_k \in K$ such that $w_{o\hat{K}}(r_k) \in \phi^+$, but $w_{o\hat{K}}(r_k) \neq r_i$ for any $r_i \in \prod_K$. Then $w_k w_{o\hat{K}} \in Y_K$, for if $r_j \in \prod_K$, $w_k w_{o\hat{K}}(r_j) = w_k(s) \in \phi^+$ where $s \in \phi^+ - \{r_k\}$, and if $r_j \in \prod_{\hat{K}}$, then $w_k w_{o\hat{K}}(r_j) = -w_k(s)$, where $s \in \prod_{\hat{K}}$, and so $w_k w_{o\hat{K}}(r_j) \in \phi^-$. Hence $|Y_K| > 1$ and so $|Y_{\hat{K}}| > 1$ by (1.3.6).

(b) Suppose that for all $r_k \in \prod_K$, $w_{o\hat{K}}(r_k) = r_k$. Then $w_{o\hat{K}}$ is a product of reflections each of which fix r_k, and $w_{o\hat{K}} w_k = w_k w_{o\hat{K}}$ for all $w_k \in K$. Further, $w_j w_k = w_k w_j$ for all $r_j \in \prod_{\hat{K}}$, $r_k \in \prod_K$, and so K and $\hat{K}$ are mutually commuting sets. Since there is no $w_k \in R$ for which $w_k w_{o\hat{K}} \in Y_K$, then $|Y_K| = 1$, and so also $|Y_{\hat{K}}| = 1$.

(1.4) Solomon's Decomposition of the Group Algebra of a Finite Coxeter Group.

These results and their proofs can be found in Solomon [20].

Let $A = Q[W]$, the group algebra of a finite Coxeter group W over Q. Let R be a set of fundamental reflections which generate W. For each subgroup W_J of W, let $A_J = Q[W_J]$, the group algebra of W_J over Q. Define

(1.4.1)
$$e_J = \frac{1}{|W_J|} \sum_{w \in W_J} w,$$
$$o_J = \frac{1}{|W_J|} \sum_{w \in W_J} \epsilon(w) w$$

where $\epsilon: W \to \{\pm 1\}$ is a homomorphism of W such that $\epsilon(w_i) = -1$ for all $w_i \in R$. (ϵ is called the alternating character of W, and $\epsilon(w) = (-1)^{l(w)}$ for all $w \in W$). e_J and o_J

are idempotents in the centre of A_J.

(1.4.2) <u>THEOREM</u>: Let (W,R) be a finite Coxeter system, and let $A = Q[W]$. Then

$$A = \sum_{J\leqslant R}^{\oplus} Ae_J o_{\hat{J}},$$

where $\dim Ae_J o_{\hat{J}} = |Y_J|$, and $Ae_J o_{\hat{J}}$ has basis $\{ye_J o_{\hat{J}} : y \in Y_J\}$. Moreover, for any $K \subseteq R$,

$$Ae_K = \sum_{\substack{J \\ K\leqslant J\leqslant R}}^{\oplus} Ae_J o_{\hat{J}} e_K.$$

Further, if ϕ_K is the character of W induced from the principal character of W_K, for $K \subseteq R$, and if ϵ is the alternating character of W, then for any subset J of R we have

$$\sum_{\substack{K \\ J\leqslant K\leqslant R}} (-1)^{|K-J|}\,\phi_K = \epsilon \sum_{\substack{K \\ \hat{J}\leqslant K\leqslant R}} (-1)^{|K-\hat{J}|}\,\phi_K.$$

If $J = \emptyset$, this reduces to $\epsilon = \sum_{\substack{J \\ J\leqslant R}} (-1)^{|J|}\,\phi_J$.

<u>EXAMPLE</u>: <u>$W = W(A_2) \cong S_3$</u>, the symmetric group on 3 symbols.

$W = \langle w_1, w_2 : w_1^2 = w_2^2 = (w_1w_2)^3 = 1\rangle$.

$J = \emptyset$, $Y_\emptyset = \{w_1w_2w_1\}$, $e_\emptyset = 1$, $o_{\{w_1,w_2\}} = \frac{1}{6}(1-w_1-w_2+w_1w_2+w_2w_1 -w_1w_2w_1)$

$J = \{w_1\}$, $Y_{\{w_1\}} = \{w_2, w_1w_2\}$, $e_{\{w_1\}} = \frac{1}{2}(1+w_1)$, $o_{\{w_2\}} = \frac{1}{2}(1-w_2)$

$J = \{w_2\}$, $Y_{\{w_2\}} = \{w_1, w_2w_1\}$, $e_{\{w_2\}} = \frac{1}{2}(1+w_2)$, $o_{\{w_1\}} = \frac{1}{2}(1-w_1)$

$J = \{w_1,w_2\}$, $Y_{\{w_1,w_2\}} = \{1\}$, $e_{\{w_1,w_2\}} = \frac{1}{6}(1+w_1+w_2+w_1w_2+w_2w_1 +w_1w_2w_1)$, $o_\emptyset = 1$.

Set $A = Q[W]$, and then we have:

$A = Ae_{\emptyset}o_{\{w_1,w_2\}} \oplus Ae_{\{w_1\}}o_{\{w_2\}} \oplus Ae_{\{w_2\}}o_{\{w_1\}} \oplus Ae_{\{w_1,w_2\}}o_{\emptyset}$, where $Ae_{\emptyset}o_{\{w_1,w_2\}}$ has basis $\{\frac{1}{6}(w_1w_2w_1-w_1w_2-w_2w_1+w_1+w_2-1)\}$, $Ae_{\{w_1\}}o_{\{w_2\}}$ has basis $\{\frac{1}{4}(1-w_2-w_2w_1+w_1w_2w_1), \frac{1}{4}(w_1-w_1w_2+w_2w_1-w_1w_2w_1)\}$, $Ae_{\{w_2\}}o_{\{w_1\}}$ has basis $\{\frac{1}{4}(1-w_1-w_1w_2+w_1w_2w_1), \frac{1}{4}(w_2-w_2w_1+w_1w_2-w_1w_2w_1)\}$, and $Ae_{\{w_1,w_2\}}o_{\emptyset}$ has basis $\{\frac{1}{6}(1+w_1+w_2+w_1w_2+w_2w_1+w_1w_2w_1)\}$.

(1.4.3) THEOREM: For any $J \subseteq R$, $Ae_Jo_{\hat{J}}$ and $Ao_{\hat{J}}e_J$ are isomorphic A-modules.

(1.4.4) Inversion Formula.

Let R be a finite set and let f be a function which has for domain the set of all subsets J of R and which takes values in some additive abelian group. If

$$g(K) = \sum_{K \leq J \leq R} f(J),$$

then
$$f(J) = \sum_{J \leq K \leq R} (-1)^{|K-J|} g(K)$$

Chapter 2: GROUPS WITH (B,N) PAIRS.

(2.1) Groups with (B,N) Pairs.

(2.1.1) Definition: A group G is said to have a (B,N) pair (G,B,N,R) if there exist subgroups B and N of G satisfying the following axioms:

(1) $G = \langle B, N \rangle$

(2) $H = B \cap N \trianglelefteq N$

(3) The group $W = \frac{N}{H}$ is finite, and is generated by a set of involutions $R = \{w_1, \ldots w_n\}$.

(4) For all $w_i \in R$ and $w \in W$,

$$w_i B w \subseteq BwB \cup Bw_i wB$$

(5) For all $w_i \in R$, $w_i B w_i \neq B$.

W is called the Weyl group of the (B,N) pair. The natural homomorphism from N to W will be denoted by θ. Since elements $w \in W$ belong to $\frac{N}{H}$, and $H \trianglelefteq N$, $H \leqslant B$, the sets wH, wB or Bw, defined as nH, nB or Bn for any $n \in N$ with $\theta(n) = w$, are well-defined in G.

(2.1.2) BRUHAT'S THEOREM (Tits [25]): Let G be a group with a (B,N) pair, with Weyl group W. Then

$$G = \bigcup_{w \in W} BwB$$

and $BwB = Bw'B$ if and only if $w = w'$ for $w, w' \in W$.

(2.1.3) THEOREM (Matsumoto [17]): Let G be a group with a (B,N) pair (G,B,N,R), with Weyl group W. Then (W,R) is a Coxeter system.

Hence by (1.2.2c), W is isomorphic to the Weyl group $W(\phi)$ of a system of roots ϕ, in such a way that R corresponds to the set of fundamental reflections for a simple system $\prod$ of ϕ. We now identify W with $W(\phi)$, and R with the set of fundamental reflections in $W(\phi)$ for the simple system $\prod$ of ϕ, and say ϕ is the root system corresponding to W.

<u>EXAMPLE</u>: Let $G = Gl_n(q)$, the group of non-singular $n \times n$ matrices over the field GF(q). Let B be the subgroup of upper triangular matrices, and N the subgroup of monomial matrices. Then $B \cap N = H$, the subgroup of diagonal matrices. $W = \frac{N}{H} \cong S_n$, the symmetric group on n symbols.

(2.1.4) <u>Definition</u>: A Borel subgroup of G is a subgroup conjugate to B. A parabolic subgroup of G is a subgroup conjugate to some subgroup $G_J = BW_JB$, where $J \subsetneq R$.

(2.1.5) <u>LEMMA</u> (Tits [25]): (1) The subgroups G_J, for $J \subsetneq R$, are the only subgroups of G which contain B.

(2) Two different parabolic subgroups which contain a common Borel subgroup are not conjugate.

(3) The normaliser of a parabolic subgroup is itself.

(2.1.6) <u>LEMMA</u>: If G has a (B,N) pair (G,B,N,R), then for all $J \subsetneq R$, G_J has a (B,N) pair (G_J,B,N_J,J), where N_J is the inverse image of W_J under the natural homomorphism θ: $N \to W$.

(2.1.7) <u>LEMMA</u>: Let G be a finite group with a (B,N) pair (G,B,N,R) and Weyl group W. Then for all $w_i \in R$, for all $w \in W$,
$$Bw_iBwB = \begin{cases} Bw_iwB \text{ if } l(w_iw) = l(w) + 1 \\ BwB \cup Bw_iwB \text{ if } l(w_iw) = l(w) - 1. \end{cases}$$

(2.1.8) <u>THEOREM</u> (Curtis [8]): Let G be a finite group with a (B,N) pair (G,B,N,R), and let W be the Weyl group of G. Then there is a bijection between the family of parabolic subgroups G_J of G and the family of parabolic subgroups W_J of W. The subgroup $G_J = BW_JB$ corresponds to W_J. Further, there exists a one-one correspondence between the (G_J,G_K)-cosets of G and the (W_J,W_K)-cosets of W.

(2.1.9) <u>THEOREM</u> (Feit-Higman [13]): If G is a group with a (B,N) pair whose Weyl group is a dihedral group of order 2n, then n = 2,3,4,6 or 8.

(2.1.10) <u>Definition</u>: A (B,N) pair (G,B,N,R) is saturated if $B \cap N = \bigcap_{n \in N} n^{-1}Bn$.

(2.1.11) <u>LEMMA</u>: If G is a group with a (B,N) pair (G,B,N,R), then G has a saturated (B,N) pair (G,B,N',R), with the same subgroup B and Weyl group W. (<u>Proof</u>: see Richen [18].)

(2.2) Groups with split (B,N) pairs.

(2.2.1) Definition: A group G has a split (B,N) pair of rank n and characteristic p if

(1) G has a (B,N) pair (G,B,N,R) of rank n (i.e. $|R| = n$).

(2) $B = UH$ where U is a normal subgroup of B and a p-group, and H is an abelian p'-subgroup of B.

(3) $H = \bigcap_{n \in N} B^n$.

We write (G,B,N,R,U) for the split (B,N) pair of G.

For each $w \in W$, choose $n_w \in N$ such that $\theta(n_w) = w$. Write $n_i = n_{w_i}$ for $i = 1,..n$, and $n_o = n_{w_o}$. Define

$$(2.2.2) \begin{cases} V = U^{n_o} \\ X_i = U \cap V^{n_i} \\ X_{-i} = X_i^{n_i} = U^{n_i} \cap U^{n_o} \\ U_i = U \cap U^{n_i} \\ U_w^+ = U \cap U^{n_w} \\ U_w^- = U \cap V^{n_w} \end{cases}$$

Since H normalises U, these definitions are independent of the choice of coset representative. Proofs of the following results in this section can be found in either Carter-Lusztig [4] or Richen [18].

(2.2.3) THEOREM: Let G be a finite group with a split (B,N) pair (G,B,N,R,U), with Weyl group W, root system ϕ and simple system Π. Then:

(1) $r_i \in \phi_w^+$ implies $U_{ww_i}^- = X_i (U_w^-)^{n_i}$ and $X_i \cap (U_w^-)^{n_i} = 1$.

(2) $r_i \in \Phi_w^-$ implies $U_w^- = X_i(U_{ww_i}^-)^{n_i}$ and $X_i \cap (U_{ww_i}^-)^{n_i} = 1$.

(3) For all $w \in W$, $U = U_w^+ U_w^- = U_w^- U_w^+$ with $U_w^+ \cap U_w^- = 1$.

In particular, $U = X_i U_i = U_i X_i$, $U_i \cap X_i = 1$ for all $i = 1,..n$.

(4) $G = \bigcup_{w \in W} Bn_w U_w^-$

(5) $B \cap V = 1$.

(6) Let $n \in N$ and $\theta(n) = w$. Then $X_i^n \subseteq U$ if $l(w_i w) = l(w) + 1$

(7) Suppose $r_i, r_j \in \prod$ with $w(r_i) = r_j$. Let $n \in N$ satisfy $\theta(n) = w$. Then $X_i^{n^{-1}} = X_j$.

(8) Let $w = w_{i_1}..w_{i_k}$ where $w_{i_j} \in R$ for $j = 1,..k$, and $l(w) = k$. Then $U_w^- = X_{i_k}(X_{i_{k-1}})^{n_{i_k}}..(X_{i_1})^{n_{i_2}..n_{i_k}}$.

(2.2.4) <u>COROLLARY</u>: Each element $g \in G$ has a unique expression of the form $g = bn_w u_w$, where $b \in B$, $w \in W$ and $u_w \in U_w^-$.

(2.2.5) <u>COROLLARY</u>: U is a p-Sylow subgroup of G.

(2.2.6) <u>Definition</u>: Let $H_i = \langle X_i, X_{-i} \rangle \cap H$.

(2.2.7) <u>LEMMA</u> : (1) The coset representative n_i can be chosen in $\langle X_i, X_{-i} \rangle$.

(2) If $r_i, r_j \in \prod$ with $w(r_i) = r_j$, let $n \in N$ satisfy $\theta(n) = w$; then $H_i^{n^{-1}} = H_j$.

(3) $\langle X_i, X_{-i} \rangle = X_i H_i \cup X_i H_i n_i X_i$.

(4) Let $x_i \in X_i - 1$. Then $n_i^{-1} x_i n_i \in X_i H_i n_i X_i$.

(2.2.8) <u>Definition</u> (Structure equation): For each $x_i \in X_i - 1$

write $n_i^{-1}x_in_i = f_i(x_i)h_i(x_i)n_ig_i(x_i)$ where $f_i(x_i)$, $g_i(x_i) \in X_i$, and $h_i(x_i) \in H_i$.

(2.2.9) THEOREM: Let Σ be the set of W-conjugates of the X_i's, $1 \leq i \leq n$. Then (W,Σ) is a permutation group under

$$w : w'X_iw'^{-1} \to ww'X_iw'^{-1}w^{-1}.$$

and $X_i^{w^{-1}} \to w(r_i)$ defines an isomorphism $(W,\Sigma) \cong (W,\phi)$.

(2.3) Irreducible Representations of Finite Groups with Split (B,N) Pairs.

The results and their proofs in this section can be found in Curtis [9], Richen [18] and Carter-Lusztig [4].

Let K be a splitting field for H of characteristic p, and let G be a finite group with a split (B,N) pair (G,B,N,R,U) of rank n and characteristic p.

(2.3.1) LEMMA: Let M be an arbitrary left KG-module. Then M contains a one-dimensional B-invariant subspace.

(2.3.2) Definition: An element $m \neq 0$ in a left KG-module M is called a weight element of weight $(\mathcal{X};\mu_1,..\mu_n)$ where $\mathcal{X} : B \to K^* = K-\{1\}$ is a homomorphism, and the $\mu_i \in K$, provided that

$$bm = \mathcal{X}(b)m \text{ for all } b \in B$$

$$S_im = \mu_im \quad \text{for all } 1 \leq i \leq n$$

where $S_i = \sum_{u \in X_i} un_i$ for $i = 1,..n$.

(2.3.3) <u>THEOREM</u>: Let G be a group with a split (B,N) pair of rank n and characteristic p, and K a field as above. Then:

(a) Every left KG-module contains a weight element.

(b) If $m \in M$ is a weight element, then $KGm = KVm$.

(c) Each irreducible left KG-module contains a unique line fixed by U, and hence a unique line fixed by B.

(d) If M_1 and M_2 are irreducible modules containing weight elements of the same weight, then $M_1 \cong M_2$. Conversely, each irreducible module determines a unique weight.

(2.3.4) <u>LEMMA</u>: Let G and K be as above. Let $(\mathcal{X};\mu_1,..\mu_n)$ be the weight of an irreducible KG-module. Then $\mu_i = 0$ or -1, and $\mu_i \neq 0$ implies $\mathcal{X}/_{H_i} = 1$.

(2.3.5) <u>THEOREM</u>: Let G be a group with a split (B,N) pair (G,B,N,R,U) of rank n and characteristic p, and let K be an algebraically closed field of characteristic p. Let $\mathcal{X} : B \rightarrow K^*$ be a homomorphism, and let $\mu_1,..\mu_n$ be elements of K such that $\mu_i = 0$ or -1. Then $(\mathcal{X};\mu_1,..\mu_n)$ is the weight of an irreducible KG-module if and only if $\mathcal{X}/_{H_i} = 1$ whenever $\mu_i \neq 0$.

(2.3.6) <u>LEMMA</u>: Every irreducible left KG-module has dimension less than $|U|$, except for the irreducible module of weight $(1_B;-1,-1,..-1)$, whose dimension is $|U|$.

(2.4) <u>Construction of the Irreducible Modules</u>.

(2.4.1) <u>THEOREM</u> (Richen [18]): If G is a finite group, H any subgroup and K any field, then every irreducible KG-module is contained in a module induced from an irreducible KH-module.

Let G be a finite group with a split (B,N) pair of rank n and characteristic p. Let K be any field. By the above theorem, we can restrict our search for the irreducible KG-modules by considering KG-modules induced from irreducible representations of B. If M is the principal KU-module, $KG \otimes_{KU} M$ is the KG-module induced from M. It is convenient to describe this module as a space of functions on the right cosets of U in G: let $\mathcal{F}$ be the set of functions $f: {}^G/U \to K$. $\mathcal{F}$ can be made into a left KG-module by defining $xf \in \mathcal{F}$, given $x \in G$, $f \in \mathcal{F}$, by

$$(xf)(Ug) = f(Ugx) \text{ for all } Ug \in {}^G/U.$$

(2.4.2) <u>LEMMA</u>: $\mathcal{F}$ is isomorphic, as left KG-module, to $KG \otimes_{KU} M$.

The results given in the rest of this section are proved in Carter-Lusztig [4].

Let χ be a one-dimensional representation of H with values in K. Define the subspace $\mathcal{F}_\chi$ of $\mathcal{F}$ by:

(2.4.3) $\mathcal{F}_\chi = \{f \in \mathcal{F}: f(Uhg) = \chi(h)f(Ug) \text{ for all } h \in H, g \in G\}$

$\mathcal{F}_\chi$ is a KG-submodule of $\mathcal{F}$. Now the linear character χ of H can be extended to a linear character of B with U in the kernel. Let M_{χ} be the one-dimensional left KB-module affording the character χ.

(2.4.4) LEMMA: $\mathcal{F}_{\chi}$ is isomorphic, as left KG-module, to $KG \otimes_{KB} M_{\chi}$.

(2.4.5) LEMMA: Suppose the characteristic of K is prime to $|H|$ and that K is a splitting field for H. Then $\mathcal{F} = \sum_{\chi}^{\oplus} \mathcal{F}_{\chi}$, summed over all one-dimensional representations χ of H.

Let $\phi_{\chi} \in \mathcal{F}$ be the function taking values

$$\phi_{\chi}(Uh) = \chi(h)$$

$$\phi_{\chi}(Ug) = 0 \text{ if } g \notin B.$$

Then $\phi_{\chi} \in \mathcal{F}_{\chi}$ and ϕ_{χ} generates the left KG-module $\mathcal{F}_{\chi}$.

(2.4.7) Definition: Given a linear character χ of H and an element $w \in W$, define $w(\chi)$ to be the linear character of H given by $(w(\chi))(h) = \chi(h^w)$ for all $h \in H$.

(2.4.8) LEMMA: If $w_i(\chi) \neq \chi$, then $\sum_{x_i \in X_i - \{1\}} w_i(\chi)(h_i(x_i)) = 0$, where the h_i is given in (2.2.8). Also, $\sum_{x_i \in X_i - \{1\}} \chi(h_i(x_i)) = 0$.

(2.4.9) Definition: For each $f \in \mathcal{F}$, define $T_n f \in \mathcal{F}$ by:

$$(T_n f)(Ug) = \sum_{Ug' \subseteq Un^{-1}Ug} f(Ug').$$

(2.4.10) PROPOSITION: (1) T_n is an endomorphism of the KG-module $\mathcal{F}$, and transforms the submodule $\mathcal{F}_{\chi}$ into $\mathcal{F}_{w(\chi)}$, where $\theta(n) = w$.

(2) The U-invariant functions in $\mathcal{F}_{\chi}$

form a subspace of dimension $|W|$. The functions $T_{n_w}\phi_{w^{-1}}(X)$ for all $w \in W$ form a basis for this subspace.

(3) Let $n, n' \in N$, with $\theta(n) = w$, $\theta(n') = w'$. Then $T_n T_{n'} = T_{nn'}$ provided $l(ww') = l(w) + l(w')$.

Write $T_i = T_{n_i}$ for $i = 1,..n$. The elements $T_1,..T_n$ are called the fundamental endomorphisms of $\mathcal{F}$.

Now assume K is a field of characteristic p, which is a splitting field for H.

(2.4.11) <u>LEMMA</u>: Let χ be a linear K-character of H. If $\chi/_{H_i}=1$, then $w_i(\chi) = \chi$. If $\chi/_{H_i}\neq 1$, then $\sum_{x_i \in X_i - \{1\}} \chi(h_i(x_i)) = 0$.
We thus have $T_i^2 = \begin{cases} 0 \text{ if } \chi/_{H_i} \neq 1 \text{ on } \mathcal{F}_\chi \\ -T_i \text{ if } \chi/_{H_i} = 1 \text{ on } \mathcal{F}_\chi. \end{cases}$

(2.4.12) <u>Definition</u>: For each linear K-character χ of H, define

$$J_0(\chi) = \{w_i \in R: \chi/_{H_i}=1\}$$
$$= \{w_i \in R: T_i^2 = -T_i \text{ on } \mathcal{F}_\chi\}.$$

For each subset J of $J_0(\chi)$ and each element $w \in W$, define a KG-homomorphism $\theta_w^J : \mathcal{F}_\chi \rightarrow \mathcal{F}_{w(\chi)}$ as follows: write $w = w_{j_k}..w_{j_1}$ where $l(w)=k$, and write $\theta_w^J = \theta_{j_k}..\theta_{j_1}$ where

$\theta_{j_i}: \mathcal{F}_{w_{j_{i-1}}..w_{j_1}(\chi)} \rightarrow \mathcal{F}_{w_{j_i}..w_{j_1}(\chi)}$ is defined by

$$\theta_{j_i} = \begin{cases} T_{j_i} & \text{if } w_{j_1}..w_{j_{i-1}}(r_{j_i}) \notin \Phi_J \\ 1+T_{j_i} & \text{if } w_{j_1}..w_{j_{i-1}}(r_{j_i}) \in \Phi_J \end{cases}$$

In the latter case, $T_{j_i}^2 = -T_{j_i}$ on $\mathcal{F}_{w_{j_{i-1}}..w_{j_1}(\chi)}$.

Now, θ_w^J is a non-zero KG-homomorphism from $\mathcal{F}_\chi$

into $\mathcal{F}_{w(\mathcal{X})}$, and is determined to within a scalar multiple by w and J. We now restrict attention to the homomorphism $\Theta^J_{w_o} : \mathcal{F}_{\mathcal{X}} \to \mathcal{F}_{w_o(\mathcal{X})}$.

(2.4.13) Definition: For each subset $J \subseteq R$, define $\bar{J}$ by

$$\bar{J} = \{w_i \in R : -w_o(r_i) \in \prod_J\}.$$

$\bar{J}$ is the image of J under the opposition involution w_o.

(2.4.14) PROPOSITION: The subspace $\Theta^J_{w_o} \mathcal{F}_{\mathcal{X}}$ of $\mathcal{F}_{w_o(\mathcal{X})}$ lies in an eigenspace of the map $T_i : \mathcal{F}_{w_o(\mathcal{X})} \to \mathcal{F}_{w_i w_o(\mathcal{X})}$ for each i. The eigenvalue of T_i on $\Theta^J_{w_o} \mathcal{F}_{\mathcal{X}}$ is given by:

0 if $w_i \in \bar{J}$

-1 if $w_i \notin \bar{J}$, $w_i \in \overline{J_o(\mathcal{X})}$

0 if $w_i \notin \overline{J_o(\mathcal{X})}$.

(2.4.15) COROLLARY: $\Theta^J_{w_o} \phi_{\mathcal{X}}$ is a U-invariant vector in $\mathcal{F}_{w_o(\mathcal{X})}$ which is an eigenvector for each T_i.

(2.4.16) Definition: Define $f^J_{\mathcal{X}} = \Theta^J_{w_o} \phi_{\mathcal{X}}$. $f^J_{\mathcal{X}}$ is determined by J and $\mathcal{X}$ to within a non-zero scalar multiple.

(2.4.17) PROPOSITION: $S_i f^J_{\mathcal{X}} = \mu_i f^J_{\mathcal{X}}$ where $\mu_i = \begin{cases} 0 \text{ if } w_i \in J \\ -1 \text{ if } w_i \in J_o(\mathcal{X}) - J \\ 0 \text{ if } w_i \notin J_o(\mathcal{X}) \end{cases}$

(2.4.18) THEOREM: The module $\Theta^J_{w_o} \mathcal{F}_{\mathcal{X}}$ is an irreducible KG-submodule of $\mathcal{F}_{w_o(\mathcal{X})}$. It has a unique one-dimensional U-invariant subspace, and this subspace is spanned by $f^J_{\mathcal{X}}$.

The stabiliser of this one-dimensional subspace is the parabolic subgroup G_J of G. The one-dimensional representation of B on this subspace is $\mathcal{X}$. The irreducible modules $\theta^J_{w_o} \mathcal{J}_{\mathcal{X}}$ for distinct pairs $(J,\mathcal{X})$ are not isomorphic.

(2.4.19) LEMMA: Every irreducible KG-module is isomorphic to a submodule of $\mathcal{J}_{\mathcal{X}}$ for some $\mathcal{X}$.

(2.4.20) THEOREM: The modules $\theta^J_{w_o} \mathcal{J}_{\mathcal{X}}$ for $J \subseteq J_o(\mathcal{X})$ are the only irreducible submodules of $\mathcal{J}_{w_o(\mathcal{X})}$. There is a natural bijection between the isomorphism classes of irreducible KG-modules and the pairs $(J,\mathcal{X})$ with $J \subseteq J_o(\mathcal{X})$.

Note that the irreducible KG-modules are absolutely irreducible since the construction and proof of irreducibility of the modules remains essentially the same when the field K is replaced by its algebraic closure.

Denote the irreducible module of weight $(\mathcal{X};\mu_1,..\mu_n)$ by $M_{(\mathcal{X};\mu_1,..\mu_n)}$.

(2.4.21) THEOREM: Let $M = KGm$, where $m \in M$ is a weight element of weight $(\mathcal{X};\mu_1,..\mu_n)$. Then M has a unique maximal submodule M', and $M/M' \cong M_{(\mathcal{X};\mu_1,..\mu_n)}$.

Proof: $M_{(\mathcal{X};\mu_1,..\mu_n)}$ is generated as KG-module by a weight element v. Define a KG-homomorphism $\sigma: M \to M_{(\mathcal{X};\mu_1,..\mu_n)}$ by $\sigma(m) = v$. σ is a well-defined KG-homomorphism, as m is a weight element of the same weight as v, and is clearly onto.

Let $M' = \ker \sigma$. Then $M/M' \cong M_{(\mathcal{X};\mu_1,..\mu_n)}$.

Let M_1 be a maximal submodule of M. Then there is a KG-homomorphism $\psi : M \to M_{(\mathcal{X}';\mu_1',..\mu_n')}$ with kernel M_1. If $\psi(m) = 0$, since $M = KGm$, then we must have $\psi = 0$. But we have supposed $\psi \neq 0$. So $\psi(m) \neq 0$, and as ψ is a KG-homomorphism, $\psi(m)$ is a weight element of $M_{(\mathcal{X}';\mu_1',..\mu_n')}$ of weight $(\mathcal{X};\mu_1,..\mu_n)$. By (2.4.18), we must have $\mathcal{X} = \mathcal{X}'$, and $\mu_i = \mu_i'$ for each i. Then ψ is a scalar multiple of σ, so $\ker \sigma = \ker \psi$, i.e. $M_1 = M'$.

Remark: Unfortunately, nothing seems to be known about the composition factors of M'.

(2.4.22) NOTE: Let G(p) be a finite Chevalley group, and H a maximal torus. Then the irreducible representations of H over the field GF(p) are the functions

$$\mathcal{X}_{a_1,..a_n} : H \to GF(p)$$

given by $(h_1,..h_n) \to h_1^{a_1}h_2^{a_2}..h_n^{a_n}$, where $0 \leq a_i \leq p-1$ for all i. The representation theory of algebraic groups leads one to associate the point $(a_1,..a_n)$, $0 \leq a_i \leq p-1$, with the pair $(\mathcal{X},J)$ where $\mathcal{X} = \mathcal{X}_{a_1,..a_n}$ and $w_i \in J$ if and only if the i-th co-ordinate of $(a_1,..a_n)$ is 0. So $(0,..0)$ is associated with the pair $(\mathcal{X}_{0,..0},\{w_1,..w_n\})$ giving the unit representation, and $(p-1,..p-1)$ is associated with the pair $(\mathcal{X}_{0,..0},\emptyset)$ giving the Steinberg representation.

Chapter 3: HECKE ALGEBRAS AND THE GENERIC RING.

(3.1) The Hecke Algebra.

We will define the Hecke algebra of a group G with respect to a subgroup B as in Bourbaki [1] (pages 54-5, exercises 22, 23, 24).

Let B be a subgroup of a group G. Suppose each double coset BgB is a finite union of right cosets of B in G. Let k be a commutative ring. Let ${}^G/B = \{Bg : g \in G\}$, and $B\backslash{}^G/B = \{BgB : g \in G\}$. For $t \in {}^G/B$, let b_t denote the map from G to k defined by

$$b_t(g) = \begin{cases} 1 \text{ if } g \in t \\ 0 \text{ if } g \notin t \end{cases}$$

for all $g \in G$. For $t \in B\backslash{}^G/B$, let a_t denote the map from G to k defined by

$$a_t(g) = \begin{cases} 1 \text{ if } g \in t \\ 0 \text{ if } g \notin t \end{cases}$$

for all $g \in G$. Let L be the k-module generated by the b_t for $t \in {}^G/B$, and H the k-module generated by the a_t for $t \in B\backslash{}^G/B$. For $t, t' \in B\backslash{}^G/B$, define

$$(3.1.1) \qquad a_t * a_{t'} = \sum_{t''} m(t,t';t'')a_{t''}$$

where $m(t,t';t'')$ = the number of right cosets of B contained in $t' \cap t^{-1}x$ for any $x \in t''$, and extend by linearity to H. This gives H a k-algebra structure, admitting a_B as unit element. H is called the Hecke algebra of G with respect to B, and written $H_k(G,B)$.

L can be made into a left H-module, by defining for each $t \in B\backslash^G/B$ and each $t' \in {}^G/B$:

(3.1.2) $$a_t * b_{t'} = \sum_{t'' \subseteq tt'} b_{t''}$$

and extending by linearity to an action of H on L.

G operates on L in the following way: for $g \in G$, $t \in {}^G/B$, define $(gb_t) \in L$ as

(3.1.3) $$gb_t = b_{tg^{-1}}$$

(3.1.4) THEOREM: The action of H on L defines an isomorphism between H and the ring of kG-endomorphisms $En_{kG}(L)$ of the left kG-module L.

Remark: It is also true that

$$m(t,t';t'') = \text{the number of left cosets of B in } t \cap x(t')^{-1}$$
$$\text{for all } x \in t''.$$

Now suppose G is a finite group with a (B,N) pair (G,B,N,R). Then, for all $w_i \in R$, the double coset Bw_iB is the union of a finite number of right cosets of B. Hence for all $w \in W$, where W is the Weyl group of G, BwB is the union of a finite number of right cosets of B. Thus we can form the Hecke algebra $H_k(G,B)$ for any commutative ring k.

Since each $t \in B\backslash^G/B$ is of the form BwB, for some $w \in W$, we write a_w for a_{BwB}. Then the map a_w: $G \to k$ satisfies $$a_w(g) = \begin{cases} 1 \text{ if } g \in BwB \\ 0 \text{ if } g \notin BwB. \end{cases}$$

The $\{a_w : w \in W\}$ form a k-basis for $H_k(G,B)$, and multiplication is given by:

$$a_w * a_{w'} = \sum_{w'' \in W} m(w,w';w'')a_{w''}$$

where $m(w,w';w'')$ = the number of right cosets of B in $Bw'B \cap Bw^{-1}Bw''b$ for any $b \in B$.

(3.1.5) <u>Definition</u>: For any $w \in W$, $w \neq 1$, define

$$q_w = |B:B \cap B^w|$$

= the number of right cosets of B in BwB.

Define $q_1 = 1$.

(3.1.6) <u>LEMMA</u>: For any $w \in W$, $w \neq 1$, let $w = w_{i_1} \cdots w_{i_s}$ be a reduced expression for w with each $w_{i_j} \in R$ for $1 \leq j \leq s$. Then $q_w = q_{w_{i_1}} \cdots q_{w_{i_s}}$, and q_w is independent of the reduced expression for w. In particular, if $w_i, w_j \in R$ are conjugate in W, then $q_{w_i} = q_{w_j}$.

<u>Proof</u>: We prove $q_w = q_{w_{i_1}} \cdots q_{w_{i_s}}$ by induction on $l(w)$. It is obviously true if $l(w) = 0$ or 1. So suppose $l(w) > 1$. Let $w' = w_{i_2} \cdots w_{i_s}$; then $w = w_{i_1}w'$ and $l(w) = l(w')+1$. By induction, $q_{w'} = q_{w_{i_2}} \cdots q_{w_{i_s}}$. Now let $b_1, \ldots b_r, b'_1, \ldots b'_t$ be elements of B such that $\{Bw'b_i\}_{i=1}^{r}$ is precisely the set of cosets of B in $Bw'B$, and $\{Bw_{i_1}b'_j\}_{j=1}^{t}$ is precisely the set of cosets of B in $Bw_{i_1}B$. We show that $\{Bw_{i_1}b'_jw'b_i : 1 \leq i \leq r, 1 \leq j \leq t\}$ is precisely the set of cosets of B in BwB.

Let Bwb be a coset of B in BwB. Then $Bw'b = Bw'b_i$

for some i, $1 \leq i \leq r$. So there exists $b' \in B$ such that $w'b = b'w'b_i$. Then $Bwb = Bw_{i_1}w'b = Bw_{i_1}b'w'b_i$. Now for some j, $1 \leq j \leq t$, we have $Bw_{i_1}b' = Bw_{i_1}b'_j$, and so $Bwb = Bw_{i_1}b'_jw'b_i$. Hence $q_w \leq q_{w_{i_1}}q_{w'}$. Conversely, since $l(w_{i_1}w') = l(w')+1$, each coset $Bw_{i_1}b'_jw'b_i \subseteq Bw_{i_1}w'B = BwB$.

It remains to show that if $Bw_{i_1}b'_jw'b_i = Bw_{i_1}b'_lw'b_u$, for some $j,l \in \{1,..t\}$, $i,u \in \{1,..r\}$, then $b_i = b_u$ and $b'_j = b'_l$. Suppose $b_i = b_u$. Then $Bw_{i_1}b'_j = Bw_{i_1}b'_l$, so $j=l$. So suppose $i \neq u$; there exists $b \in B$ with $bw_{i_1}b'_jw'b_i = w_{i_1}b'_lw'b_u$. Then $w_{i_1}bw_{i_1}b'_jw'b_i = b'_lw'b_u$. But $w_{i_1}bw_{i_1} \in B \cup Bw_{i_1}B$, so $w_{i_1}bw_{i_1} = b'$ for some $b' \in B$, or $w_{i_1}bw_{i_1} = b'w_{i_1}b''$ for some $b',b'' \in B$. In the first case we have $b'b'_jw'b_i = b'_lw'b_u$ and so we must have $i=u$, contrary to assumption. In the second case, $b'w_{i_1}b''b'_jw'b_i = b'_lw'b_u$, and as $l(w_{i_1}w') > l(w')$, $b'w_{i_1}b''b'_jw'b_i \in Bw_{i_1}w'B = BwB$. But $b'_lw'b_u \in Bw'B$ - this is impossible, as if $w \neq w'$, then $BwB \cap Bw'B = \emptyset$. Thus, $q_w = q_{w_{i_1}}q_{w'} = q_{w_{i_1}} .. q_{w_{i_s}}$.

The rest of the lemma follows from (1.2.3), and by noting that if w_iw_j has odd order n_{ij}, where $w_i, w_j \in R$, then $(q_{w_i}q_{w_j}q_{w_i}..)_{n_{ij}} = (q_{w_j}q_{w_i}q_{w_j}..)_{n_{ij}}$ as $(w_iw_jw_i..)_{n_{ij}}$ and $(w_jw_iw_j..)_{n_{ij}}$ are reduced expressions in W which are equal.

Let $w_i \in R$, and $w \in W$. Evaluating various $m(w,w';w'')$'s we get the following equations:

(3.1.7) $a_{w_i} * a_{w_i} = q_{w_i} a_1 + (q_{w_i}-1)a_{w_i}$

(3.1.8) $a_{w_i} * a_w = \begin{cases} a_{w_i w} & \text{if } l(w_i w) = l(w)+1 \\ (q_{w_i}-1)a_w + q_{w_i} a_{w_i w} & \text{if } l(w_i w) = l(w)-1 \end{cases}$

We then get the following two theorems about the structure of $H_k(G,B)$:

(3.1.9) THEOREM: $H_k(G,B)$ is the associative k-algebra with k-basis $\{a_w\}_{w \in W}$, and multiplication given by

$$a_{w_i} * a_w = \begin{cases} a_{w_i w} & \text{if } l(w_i w) = l(w)+1 \\ (q_{w_i}-1)a_w + q_{w_i} a_{w_i w} & \text{if } l(w_i w) = l(w)-1 \end{cases}$$

for all $w_i \in R$ and all $w \in W$. a_1 is the identity element.

(3.1.10) THEOREM: $H_k(G,B)$ is the associative k-algebra with identity a_1 generated by $\{a_{w_i} : w_i \in R\}$ subject to the relations:

$$a_{w_i} * a_{w_i} = (q_{w_i}-1)a_{w_i} + q_{w_i} a_1 \quad \text{for all } w_i \in R$$

$$(a_{w_i} * a_{w_j} * a_{w_i} \cdot\cdot)_{n_{ij}} = (a_{w_j} * a_{w_i} * a_{w_j} \cdot\cdot)_{n_{ij}} \quad \text{for all}$$

$w_i, w_j \in R$, $i \neq j$, where n_{ij} = order of $w_i w_j$ in W.

Now we wish to examine the left kG-module L. Let M be the principal kB-module, (i.e. M affords the trivial representation of B over k), and consider the kG-module $M^G = kG \otimes_{kB} M$ induced from M. Then the left

kG-module L is isomorphic (as left kG-module) to M^G.
Hence L affords the representation $(1_B)^G$ of G over k.
We may regard L as the set of functions from the right cosets of B in G to k; then L has k-basis $\{f_{Bg}: Bg \in {}^G/B\}$ where $f_{Bg} : {}^G/B \to k$ is given by

$$f_{Bg}(Bg') = \begin{cases} 1 \text{ if } Bg' = Bg, \\ 0 \text{ if } Bg' \neq Bg. \end{cases}$$

Further, for all $g' \in G$, for all $Bg \in {}^G/B$, we have

$$g'f_{Bg} = f_{Bg(g')^{-1}}$$

More generally, if $f \in L$, then for all $g' \in G$, for all $Bg \in {}^G/B$, we have $(g'f)(Bg) = f(Bgg')$.

By (3.1.4), $En_{kG}(L)$ is isomorphic to $H_k(G,B)$, which has k-basis $\{a_w : w \in W\}$, such that

(3.1.11) $a_w f_{Bg} = \sum_{Bg' \subseteq BwBg} f_{Bg'}$ for all $w \in W$ and all $Bg \in {}^G/B$.

(3.1.12) <u>LEMMA</u>: For any $f \in L$, and for any $w \in W$,

$$a_w f(Bg) = \sum_{Bg' \subseteq Bw^{-1}Bg} f(Bg') \text{ for all } Bg \in {}^G/B.$$

<u>Proof</u>: $a_w f_{Bx}(Bg) = \sum_{Bx' \subseteq BwBx} f_{Bx'}(Bg) = \sum_{Bg' \subseteq Bw^{-1}Bg} f_{Bx}(Bg')$.

Result now follows as every element of L is a k-linear combination of elements of the form f_{Bx} with $Bx \in {}^G/B$.

Now take $k = K$, a field of characteristic p, $p \neq 0$.
Suppose that for all $w_i \in R$, $q_{w_i} = 0$ in K. Let $L = \{f: {}^G/B \to K\}$, and then by (3.1.4) and (3.1.10) we have that $H_K(G,B) \cong En_{KG}(L)$ is the associative K-algebra with

identity a_1 generated by $\{a_{w_i}: w_i \in R\}$ subject to the relations:

$$a_{w_i}^2 = -a_{w_i} \text{ for all } w_i \in R$$

$$(a_{w_i}a_{w_j}a_{w_i}..)_{n_{ij}} = (a_{w_j}a_{w_i}a_{w_j}..)_{n_{ij}}, \text{ for all}$$

$w_i, w_j \in R$, $i \neq j$, where n_{ij} = order of $w_i w_j$ in W.

Moreover, if $w \in W$ has a reduced expression $w_{i_1}..w_{i_s}$ with all $w_{i_j} \in R$, then $a_w = a_{w_{i_1}}..a_{w_{i_s}}$, and for any $f \in L$,

$$a_w f(Bg) = \sum_{Bg' \subseteq Bw^{-1}Bg} f(Bg').$$

Such an algebra is an example of a 0-Hecke algebra, which we will define below. In Chapter 4, we will examine the structure of a 0-Hecke algebra. When k=K is a field of characteristic p, and all $q_{w_i} = 0$ in K, we will denote the Hecke algebra $H_K(G,B)$ by $H_K(0)$.

(3.1.13) <u>Definition</u>: The 0-Hecke algebra H_K over the field K of type (W,R), where (W,R) is a finite Coxeter system, is the associative algebra over K with identity a_1 generated by $\{a_{w_i}: w_i \in R\}$ subject to the relations:

(a) $a_{w_i}^2 = -a_{w_i}$ for all $w_i \in R$

(b) $(a_{w_i}a_{w_j}a_{w_i}..)_{n_{ij}} = (a_{w_j}a_{w_i}a_{w_j}..)_{n_{ij}}$ for all

$w_i, w_j \in R$, $i \neq j$, where n_{ij} = order of $w_i w_j$ in W.

For all $w \in W$, define $a_w = a_{w_{i_1}} .. a_{w_{i_s}}$, where $w = w_{i_1} .. w_{i_s}$ is a reduced expression for w in terms of the elements of R. a_w is clearly independent of the expression for w, by (1.1.12(7)) and (3.1.13(b)).

(3.1.14) PROPOSITION: Every element of H_K is a K-linear combination of elements a_w, for $w \in W$.

Proof: Consider products of the form $a_{w_{i_1}} .. a_{w_{i_t}}$. Consider the corresponding expression $w_{i_1} .. w_{i_t}$ in W. If it is reduced, then $w_{i_1} .. w_{i_t} = w$ for some $w \in W$, and $a_w = a_{w_{i_1}} .. a_{w_{i_t}}$. So suppose $w_{i_1} .. w_{i_t}$ is not reduced; then there exists an r, $1 \leq r < t$ such that $w' = w_{i_1} .. w_{i_r}$ is reduced, but $w' w_{i_{r+1}}$ is not reduced. Hence by (1.1.12(5)) there is a reduced expression for w' ending with $w_{i_{r+1}}$, say $w' = w_{j_1} .. w_{j_r}$, where $w_{j_r} = w_{i_{r+1}}$. Then

$a_{w_{i_1}} .. a_{w_{i_r}} a_{w_{i_{r+1}}} = a_{w_{j_1}} .. a_{w_{j_r}} a_{w_{i_{r+1}}} = - a_{w_{j_1}} .. a_{w_{j_r}}$ as $j_r = i_{r+1}$ and $a_{w_{j_r}}^2 = -a_{w_{j_r}}$. Continuing in this way, we show $a_{w_{i_1}} .. a_{w_{i_t}} = \pm a_{w_{k_1}} .. a_{w_{k_s}}$ for some $w = w_{k_1} .. w_{k_s}$ in W, with $l(w) = s$. Hence the result.

Now from Bourbaki [1], exercise 23, page 55, we have that $\{a_w : w \in W\}$ are linearly independent over K and so form a K-basis for H_K.

<u>Example</u>: Let $G = G(q)$ be a Chevalley group over the finite field $F = GF(q)$ of q elements, where $q = p^n$ for some prime p and positive integer n. Then G has a (B,N) pair (G,B,N,R) with Weyl group W such that for each $w_i \in R$ there is a strictly positive integer c_i such that $q_{w_i} = q^{c_i}$. Suppose K is a field of characteristic p. Then for all $w_i \in R$, $q_{w_i} = 0$ in K, and the Hecke algebra $H_K(G,B)$ is a 0-Hecke algebra.

In general, let G be a finite group with a split (B,N) pair (G,B,N,R,U) of rank n and characteristic p with Weyl group W, and let K be a field of characteristic p. Then for all $w_i \in R$, $q_{w_i} = 0$ in K, and $H_K(G,B) \cong \mathrm{En}_{KG}(I_U)$ is a 0-Hecke algebra of type (W,R) over K.

(3.2) Systems of Groups with (B,N) Pairs.

(3.2.1) Definition (Curtis, Iwahori, Kilmoyer [10]):

A system S of finite groups with (B,N) pairs of type (W,R) consists of a Coxeter system (W,R), an infinite set $\mathcal{P}$ of prime powers q, a set of positive integers $\{c_i: w_i \in R\}$, and for each $q \in \mathcal{P}$, a finite group G(q) with a (B,N) pair (G(q),B(q),N(q),R) having (W,R) as its Coxeter system, such that the following conditions are satisfied:

(1) $c_i=c_j$ for $w_i,w_j \in R$ if w_i and w_j are conjugate in W.

(2) for each group $G = G(q) \in S$,

$$\mathrm{ind}_{B(q)}w_i = |B(q):B(q) \cap B(q)^{w_i}| = q^{c_i} \text{ for all } w_i \in R.$$

Examples: $S = \{G(q) = SL_n(q)$, the group of n × n matrices of determinant 1 over the field GF(q) of q elements$\}$

S is a system of groups with (B,N) pairs of type (W,R), where $W = W(A_{n-1})$. $\mathrm{Ind}_{B(q)}w_i = q$ for all $w_i \in R$, so $c_i = 1$ for all $w_i \in R$.

More generally, each of the families of finite Chevalley groups, and twisted Chevalley groups (see Carter [3]) forms a system of (B,N) pairs.

(3.2.2) Definition: For all $w \in W$, $w \neq 1$, define

$$\mathrm{ind}_{B(q)}w = |B(q):B(q) \cap B(q)^{w}|$$

Let $\mathrm{ind}_{B(q)}1 = 1$.

(3.2.3) LEMMA: For any $w \in W$, $w \neq 1$,

$$\operatorname{ind}_{B(q)} w = q^{c(w)}$$

where $c(w) = c_{i_1} + \dots c_{i_s}$ for any reduced expression

$w = w_{i_1} \dots w_{i_s}$ of w with $w_{i_j} \in R$ for all j.

(3.2.4) Definition: Let u be an indeterminate. The characteristic function of S is the polynomial

$$\Psi(u) = \sum_{w \in W} \Psi_w(u)$$

where $\Psi_w(u) = u^{c(w)}$ for all $w \in W$, $w \neq 1$, and $\Psi_1(u) = 1$.

(3.3) The 'Classical' Hecke Algebra.

Let S be a system of finite groups with (B,N) pairs of type (W,R). Let K be a field of characteristic zero, and let $q \in \mathcal{P}$. Define in the group algebra $KG(q)$ the idempotent $b(q) = \frac{1}{|B(q)|} \sum_{x \in B(q)} x$, and the left ideal $V(q) = KG(q).b(q)$. If we regard $V(q)$ as left KG-module, we see that $V(q)$ affords a representation of $G(q)$ over K with character $(1_{B(q)})^{G(q)}$. In particular, $\dim_K V(q) = |G(q):B(q)|$. (see, for example, Curtis and Reiner [11]).

(3.3.1) Definition: Define the Hecke algebra $H_K(q) = H_K(G(q),B(q))$ as $H_K(q) = b(q).KG(q).b(q)$.

(3.3.2) LEMMA: $H_K(q)$ acts by right multiplication on $V(q)$, and if $h \in H_K(q)$, the map $\theta(h): v \to vh$ (for all $v \in V(q)$)

defines a $KG(q)$-endomorphism of $V(q)$. The map

$$\theta : H_K(q) \to En_{KG(q)}(V(q))$$

is an isomorphism of K-algebras.

The structure of the Hecke algebra $H_K(q)$ has been determined by Iwahori [16] and Matsumoto [17], and is as follows:

(3.3.3) <u>THEOREM</u>: $H_K(q)$ has K-basis $\{h_w : w \in W\}$ where

$$h_w = \mathrm{ind}_{B(q)} w.b(q)n(q)_w b(q)$$

for any $n(q)_w \in B(q)wB(q)$ for all $w \in W$. $h_1 = b(q)$ is the identity element of $H_K(q)$. For any $w \in W$, $w \neq 1$, and for any reduced expression $w = w_{i_1}..w_{i_s}$ for w, with the $w_{i_j} \in R$,

$$h_w = h_{w_{i_1}}..h_{w_{i_s}}.$$

$H_K(q)$ is generated as K-algebra with identity element h_1 by $\{h_{w_i} : w_i \in R\}$, subject to the relations:

$$h_{w_i}{}^2 = q^{c_i}h_1 + (q^{c_i}-1)h_{w_i}$$

$$(h_{w_i}h_{w_j}h_{w_i}..)_{n_{ij}} = (h_{w_j}h_{w_i}h_{w_j}..)_{n_{ij}} \text{ for all } w_i, w_j \in R,$$

$i \neq j$, where n_{ij} = order of w_iw_j in W.

(3.3.4) <u>COROLLARY</u>: For all $t,v,w \in W$ there exist polynomials $z_{t,v,w}(u) \in Z[u]$ such that

(1) for any $q \in \mathcal{P}$, $h_th_v = \sum_{w\in W} z_{t,v,w}(q)h_w$.

(2) $\sum_{w\in W} z_{t,v,w}(u)z_{w,x,y}(u) = \sum_{s\in W} z_{t,s,y}(u)z_{v,x,s}(u)$

for all $t,v,x,y \in W$.

Proof: (1) Follows from (3.3.3).

(2) Replacing u by q for any $q \in \mathcal{P}$, this equation is true by the associativity of $H_K(q)$. Since $\mathcal{P}$ is an infinite set, the equation must be an identity in u.

(3.4) The Generic Ring.

The following definition is due to J. Tits:

(3.4.1) Definition: The generic ring $A_{\mathcal{O},S}(u) = A_{\mathcal{O}}(u)$ of the system S of finite groups with (B,N) pairs of type (W,R) is the associative algebra over $\mathcal{O} = Q[u]$, where u is an indeterminate, with identity a_1 and basis $\{a_w : w \in W\}$ satisfying:

$$a_{w_i} a_w = \begin{cases} a_{w_i w} & \text{if } l(w_i w) > l(w) \\ u^{c_i} a_{w_i w} + (u^{c_i} - 1) a_w & \text{if } l(w_i w) < l(w) \end{cases}$$

for all $w \in W$, and $w_i \in R$.

Alternatively, $A_{\mathcal{O}}(u)$ is the algebra over $\mathcal{O}$ with identity a_1 and basis $\{a_w : w \in W\}$, and multiplication given by:

$$a_x a_y = \sum_{w \in W} z_{x,y,w}(u) a_w \quad \text{for all } x,y \in W$$

where $z_{x,y,w}(u) \in Z[u]$ is the polynomial determined in (3.3.4)

Let K be any extension ring of $\mathcal{O}$.

(3.4.2) Definition: The generic ring $A_K(u)$ of the system

S of finite groups with (B,N) pairs of type (W,R) is the algebra over K with identity a_1 and basis $\{a_w : w \in W\}$, and with multiplication given by

$$a_x a_y = \sum_{w \in W} z_{x,y,w}(u) a_w \text{ for all } x,y \in W.$$

Note: By (3.3.3) and (3.3.4(2)), $A_{\sigma}(u)$ and $A_K(u)$, for any extension ring K of σ, are the associative algebras over σ and K respectively generated by $\{a_{w_i} : w_i \in R\}$ subject to the relations:

$$a_{w_i}^{\,2} = u^{c_i} a_1 + (u^{c_i} - 1) a_{w_i} \text{ for all } w_i \in R, \text{ where}$$

a_1 is the identity element.

$$(a_{w_i} a_{w_j} a_{w_i} \ldots)_{n_{ij}} = (a_{w_j} a_{w_i} a_{w_j} \ldots)_{n_{ij}} \text{ for all } w_i, w_j \in R,$$

$i \neq j$, where n_{ij} = order of $w_i w_j$ in W. Further, for all $w \in W$, $a_w = a_{w_{i_1}} \ldots a_{w_{i_s}}$, where $w_{i_1} \ldots w_{i_s}$ is a reduced expression for w in terms of the elements of R.

(3.4.3) Definition: Let $f : \sigma \to Q$ be a homomorphism of commutative rings. Then define

$$A_{f,\sigma}(f(u)) = Q \otimes_{\sigma} A_{\sigma}(u)$$

where Q is viewed as a (Q,σ) bimodule by way of f:

$$axh = axf(h) \text{ for all } a,x \in Q,\ h \in \sigma.$$

$A_{f,\sigma}(f(u))$ is an associative algebra over Q, called a specialised algebra of $A_{\sigma}(u)$. It has Q-basis $\{1 \otimes a_w = a_{w,f} : w \in W\}$, and multiplication is given by:

$$a_{x,f} a_{y,f} = \sum_{w \in W} f(z_{x,y,w}(u)) a_{w,f} \text{ for all } x,y \in W.$$

Note that f induces a ring epimorphism $f':A_{\mathcal{O}}(u) \to A_{f,\mathcal{O}}(f(u))$.

EXAMPLES: (1) Let f_q be the Q-linear map $f_q:\mathcal{O} \to Q$ given by $f_q(u) = q$, where $q \in \mathcal{P}$. Then f_q induces a ring epimorphism $f_q':A_{\mathcal{O}}(u) \to A_{f_q,\mathcal{O}}(q) \cong H_Q(q)$.

(2) Let f_1 be the Q-linear map $f_1:\mathcal{O} \to Q$ given by $f_1(u) = 1$. Then f_1 induces a ring epimorphism $f_1':A_{\mathcal{O}}(u) \to A_{f_1,\mathcal{O}}(1) \cong QW$, the group algebra of W over Q.

(3) Let f_o be the Q-linear map $f_o:\mathcal{O} \to Q$ given by $f_o(u) = 0$. Then f_o induces a ring epimorphism $f_o':A_{\mathcal{O}}(u) \to A_{f_o,\mathcal{O}}(0) \cong H_Q$, the 0-Hecke algebra over Q.

(3.4.4) Definition(due to Green [14]): Let k be a subfield of C, and let $q \in k$. Then $A_k(q)$ is defined as the algebra over k with identity a_1 and k-basis $\{a_w : w \in W\}$, and with multiplication given by:

$$a_x a_y = \sum_{w \in W} z_{x,y,w}(q) a_w \quad \text{for all } x,y \in W.$$

where $z_{x,y,w}(u) \in Z[u]$ is the polynomial determined in (3.3.4).

Note: By (3.3.3) and (3.3.4(2)) $A_k(q)$ is the associative algebra over k with identity a_1, generated by $\{a_{w_i} : w_i \in R\}$ subject to the relations

$$a_{w_i}^2 = q^{c_i} a_1 + (q^{c_i} - 1) a_{w_i} \quad \text{for all } w_i \in R$$

$$(a_{w_i} a_{w_j} a_{w_i} \cdots)_{n_{ij}} = (a_{w_j} a_{w_i} a_{w_j} \cdots)_{n_{ij}} \quad \text{for all } w_i, w_j \in R,$$

$i \neq j$, where n_{ij} = order of $w_i w_j$ in W.

$a_w = a_{w_{i_1}} .. a_{w_{i_s}}$ where $w_{i_1} .. w_{i_s}$ is a reduced expression for w in terms of the elements of R.

(3.4.5) <u>LEMMA</u> (Green [14]): (1) For each $q \in \mathcal{P}$, there is a k-algebra isomorphism $s_{k,q}: A_k(q) \to H_k(q)$ which takes $a_w \to h_w$ for all $w \in W$.

(2) There is a k-algebra isomorphism $s_{k,1}: A_k(1) \to kW$ which takes $a_w \to w$ for all $w \in W$.

<u>NOTATION</u>: For the remainder of this section, we will use the following notation:

$\mathcal{O} = Q[u]$

$K_o = Q(u)$

K = a finite field extension of K_o which is a splitting field for $A_K(u)$.

I = integral closure of $\mathcal{O}$ in K

k = subfield of C.

(3.4.6) <u>THEOREM</u> (Tits - see Green [14]): $A_K(u)$ is semi-simple.

(3.4.7) <u>COROLLARY</u>: $A_{K_o}(u)$ is semi-simple, hence separable as the characteristic of K_o is zero.

(3.4.8) <u>THEOREM</u> (Tits - see Green [14]): If q is any complex number such that $q\Psi(q) \neq 0$, then $A_C(q) \cong CW$. In particular this holds for any real positive q. If q is any element of $\mathcal{P}$, then $H_C(q) \cong CW$.

We now discuss specialisations of $A_K(u)$, which are defined by Green [14] (section 4).

Definition: If P is a prime ideal of I, let K_P be the ring

$$K_P = \{{}^a/b : a \in I,\ b \in I-P\}$$

where $I-P = \{i \in I : i \notin P\}$. Then a specialisation of K with nucleus P is a ring homomorphism $f: K_P \to C$ such that $f(1) = 1$ and $\ker f = PK_P$. If $a \in K$, say $f(a)$ is defined if and only if $a \in K_P$. The range $k = f(K_P)$ of f is a subfield of C.

Let f be a specialisation of K, with nucleus P and range k. f can be extended to a ring epimorphism

$$f : A_{K_P}(u) \to A_k(q)$$

where $f(u) = q$ and $A_{K_P}(u) = \{\sum_{w\in W} s_w a_w : s_w \in K_P \text{ for all } w \in W\}$.

Then if $x = \sum_{w\in W} s_w a_w \in A_K(u)$, where $s_w \in K$ for all $w \in W$, we say $f(x)$ is defined if and only if $x \in A_{K_P}(u)$; in this case, $f(x) = \sum_{w\in W} f(s_w) a_w$.

(3.4.9) THEOREM (Green [14]): Given any specialisation f_o of K_o, whose nucleus is the prime ideal P_o of $\mathfrak{O}$, then there exists at least one specialisation f of K which extends f_o. If P is the nucleus of f, then $P_o = P \cap \mathfrak{O}$.

(3.4.10) COROLLARY (Green [14]): Given any element q of C there exists a specialisation f of K such that $f(u) = q$.

If $q \in Q$, and if P is the nucleus of f, then

$$(u-q)\sigma = P \cap \sigma.$$

Proof: There is a unique specialisation f_o of K_o such that $f_o(u) = q$, and so the existence follows from the theorem above. If $q \in Q$, the nucleus of f_o is $(u-q)\sigma$.

Suppose that for every $q \in \mathcal{P}$ we have a specialisation f_q of K such that $f_q(u) = q$. Let P_q be the nucleus of f_q. If $q,q' \in \mathcal{P}$, $q \neq q'$, then by the corollary $P_q \neq P_{q'}$.

Finally we look at some specialisations of $A_K(u)$, where K is an extension ring of σ.

(3.4.11) Let $B_o = \{{}^{g(u)}/h(u) : g(u),h(u) \in \sigma,\ u \nmid h(u)\}$.
Let $f_o : B_o \to Q$ be the Q-linear homomorphism of rings defined by $f_o(u) = 0$. Then f_o induces a ring epimorphism $f_o' : A_{B_o}(u) \to A_{f_o,B_o}(0) \cong H_Q$, the 0-Hecke algebra over Q.

(3.4.12) For any $q \in \mathcal{P}$, let $B_q = \{{}^{g(u)}/h(u) : g(u),h(u) \in \sigma,\ (u-q) \nmid h(u)\}$

Let $f_q : B_q \to Q$ be the Q-linear ring homomorphism given by $f_q(u) = q$. Then f_q induces a ring epimorphism $f_q' : A_{B_q}(u) \to A_{f_q,B_q}(q) \cong H_Q(q)$.

(3.5) The Algebra H_o.

The algebra H_o was defined by Starkey [22] (section 1.5). By (3.4.9) there exists a specialisation f_o of K with nucleus P and range k_o (where k_o is a subfield of C) such that $f_o(u) = 0$. Also, $P \cap \mathcal{O} = u\mathcal{O} = (u)$. f_o induces a ring epimorphism $f_o':A_{K_P}(u) \to A_{k_o}(0)$ given by

$f_o'(\sum_{w\in W} s_w a_w) = \sum_{w\in W} f_o(s_w)a_w$, where $s_w \in K_P$ for all $w \in W$.

(3.5.1) Definition: Let H_o be the k_o-algebra with identity a_1 and basis $\{a_w:w \in W\}$ and multiplication given by

$$a_x a_y = \sum_{w\in W} z_{x,y,w}(0)a_w \text{ for all } x,y \in W.$$

From the above we see that $H_o = A_{k_o}(0)$.

By the Iwahori-Matsumoto theorem (3.3.3), we have that H_o is generated as k_o-algebra with identity a_1 by $\{a_{w_i}:w_i \in R\}$ subject to the relations:

$$a_{w_i}^2 = -a_{w_i} \text{ for all } w_i \in R$$

$$(a_{w_i}a_{w_j}a_{w_i}\ldots)_{n_{ij}} = (a_{w_j}a_{w_i}a_{w_j}\ldots)_{n_{ij}} \text{ for all}$$

$w_i,w_j \in R$, $i\neq j$, where n_{ij} = order of w_iw_j in W.

Also, $a_w = a_{w_{i_1}}\ldots a_{w_{i_s}}$ where $w_{i_1}\ldots w_{i_s}$ is a reduced expression for w in terms of the elements of R.

Unlike the algebra $A_K(u)$, the algebra H_o is not semi-simple. However, H_o is the O-Hecke algebra H_{k_o}, whose structure we will determine in Chapter 4.

Starkey [22] has defined the decomposition numbers for H_o, and we will give the definition of these and relate them to the Cartan matrix of H_o in Chapter 5.

Chapter 4: DECOMPOSITIONS OF THE 0-HECKE ALGEBRAS.

(4.1) Introduction.

Let K be a field. Let (W,R) be a finite Coxeter system, and let $H = H_K$ be the 0-Hecke algebra of type (W,R) over K, as defined in (3.1.13). By (3.1.14) H has K-basis $\{a_w : w \in W\}$, where $a_w = a_{w_{i_1}} \ldots a_{w_{i_s}}$ if $w = w_{i_1} \ldots w_{i_s}$ with $l(w) = s$. We write $a_1 = 1$.

(4.1.1) LEMMA: For all $w_i \in R$ and all $w \in W$, we have

$$a_{w_i} a_w = \begin{cases} a_{w_i w} & \text{if } l(w_i w) = l(w)+1 \\ -a_w & \text{if } l(w_i w) = l(w)-1 \end{cases}$$

$$a_w a_{w_i} = \begin{cases} a_{ww_i} & \text{if } l(ww_i) = l(w)+1 \\ -a_w & \text{if } l(ww_i) = l(w)-1 \end{cases}$$

Proof: If $l(w_i w) = l(w)+1$, then $a_{w_i w} = a_{w_i} a_w$ by the definition of $a_{w_i w}$. Similarly, if $l(ww_i) = l(w)+1$ then $a_{ww_i} = a_w a_{w_i}$. Suppose $l(w_i w) = l(w)-1$; then there is a reduced expression for w beginning with w_i: say $w = w_i w'$ where $l(w) = l(w')+1$. Then $a_w = a_{w_i} a_{w'}$, and so $a_{w_i} a_w = a_{w_i} a_{w_i} a_{w'} = -a_{w_i} a_{w'} = -a_w$. Similarly, if $l(ww_i) = l(w)-1$, there exists a reduced expression for w ending with w_i, and in this case we get that $a_w a_{w_i} = -a_w$.

(4.1.2) COROLLARY:

(1) For all $w,w' \in W$, $a_w a_{w'} = \pm a_{w''}$ for some $w'' \in W$, with $l(w'') \geqslant \max (l(w),l(w'))$.

(2) For all $w,w' \in W$, $a_w a_{w'} = a_{ww'}$ if and only if $l(ww') = l(w)+l(w')$.

(3) For all $w,w' \in W$, $a_w a_{w'} = (-1)^{l(w')} a_w$ if and only if $w(r_i) \in \Phi^-$ for each $r_i \in \prod_J$, where $J = \{w_i \in R$: w_i occurs in some reduced expression for $w'\}$.

(4) For all $w,w' \in W$, $a_w a_{w'} = (-1)^{l(w)} a_{w'}$ if and only if $(w')^{-1}(r_i) \in \Phi^-$ for each $r_i \in \prod_J$, where $J = \{w_i \in R$: w_i occurs in some reduced expression for $w\}$.

(5) For all $w,w' \in W$, $a_w a_{w'} = \pm a_{w''}$ with $l(w'') > l(w)$, where $l(w) \geqslant l(w')$, if and only if there exists $r_i \in \prod_J$, where $J = \{w_j \in R: w_j$ occurs in some reduced expression for $w'\}$ such that $w(r_i) \in \Phi^+$.

(6) Let w_o be the unique element of maximal length in W. Then for all $w \in W$, $a_w a_{w_o} = (-1)^{l(w)} a_{w_o}$

and $\qquad a_{w_o} a_w = (-1)^{l(w)} a_{w_o}$

Example: Let $H = H(A_2)$, the 0-Hecke algebra with K-basis $\{a_w : w \in W(A_2)\}$, which is generated as K-algebra by $\{a_{w_1}, a_{w_2}\}$ subject to the relations $a_{w_i}^{\,2} = -a_{w_i}$ for $i=1,2$, and $a_{w_1}a_{w_2}a_{w_1} = a_{w_2}a_{w_1}a_{w_2}$. The multiplication table is given below: set $a_1 = 1$, $a_{w_1} = x$, $a_{w_2} = y$.

(a) \ (b)	1	x	y	xy	yx	xyx
1	(ab) 1	x	y	xy	yx	xyx
x	x	-x	xy	-xy	xyx	-xyx
y	y	yx	-y	xyx	-yx	-xyx
xy	xy	xyx	-xy	-xyx	-xyx	xyx
yx	yx	-yx	xyx	-xyx	-xyx	xyx
xyx	xyx	-xyx	-xyx	xyx	xyx	-xyx

(4.2) The Nilpotent Radical of H.

Definition: (1) The nilpotent radical N of an algebra A is the sum of all nilpotent ideals of A.

(2) The Jacobson radical J(A) of an algebra A is the intersection of all maximal ideals of A.

(4.2.1) PROPOSITION: If A has the DCC (= descending chain condition) then N = J(A) and N is nilpotent.

Proof: See, for example, Curtis and Reiner [11].

Let N be the nilpotent radical of H. Since H is a finite-dimensional algebra over the field K, H has the DCC and the ACC, and so N is also the Jacobson radical of H, and is the unique maximal nilpotent ideal of H.

(4.2.2) LEMMA: Let L be an ideal of H. Then either

(a) $a_{w_o} \in L$, or

(b) $a_{w_o} L = 0$.

Proof: By (4.1.2(6)), for all $x \in H$, $xa_{w_o} = a_{w_o}x = Ma_{w_o}$

for some $M \in K$. Hence, for all positive integers n,

$$x^n a_{w_o} = a_{w_o} x^n = M^n a_{w_o}$$

If $M \neq 0$, then $M^n \neq 0$ and so $x^n \neq 0$ for all positive integers n. So if there exists an $x \in L$ such that $xa_{w_o} = Ma_{w_o}$ for some $M \neq 0$, $M \in K$, then $a_{w_o} \in L$. Otherwise $a_{w_o}L = 0$.

(4.2.3) PROPOSITION: $(-1)^{l(w_o)} a_{w_o}$ is an idempotent in H, and $H = H(-1)^{l(w_o)} a_{w_o} \oplus H(1 - (-1)^{l(w_o)} a_{w_o})$, where $H(-1)^{l(w_o)} a_{w_o} \cong K$.

Proof: By (4.1.2(6)) we have that $(-1)^{l(w_o)} a_{w_o}$ is an idempotent in the centre of H, and so the direct sum decomposition follows by basic ring theory. The two summands are ideals of H, and since for all $x \in H$ we have $xa_{w_o} = Ma_{w_o}$, $M \in K$, it follows that $H(-1)^{l(w_o)} a_{w_o} \cong K$.

There is a natural composition series for H, consisting of (two-sided) ideals of H such that every factor is a one-dimensional H-module. This series arises as follows: list the basis elements $\{a_w : w \in W\}$ in order of increasing length of w - if $w, w' \in W$ have the same length, it does not matter in which order they occur in the list. Rename these elements $h_1, h_2, \ldots h_{|W|}$ respectively; note that $h_1 = a_1 = 1$, $h_{|W|} = a_{w_o}$. Let H_j be the ideal of H generated by $\{h_m : m \geq j\}$. H_j has K-basis $\{h_m : m \geq j\}$ and dimension $|W|-j+1$. Then

$$(4.2.4) \qquad H = H_1 > H_2 > \ldots > H_{|W|} = a_{w_o} H > 0$$

is the natural composition series of H described above. H_i/H_{i+1} is a one-dimensional H-module, with basis $h_i + H_{i+1}$, where $h_i = a_w$ for some $w \in W$. Now either $a_w^2 = (-1)^{l(w)} a_w$ or $a_w^2 \in H_{i+1}$. In the first case, the factor ring H_i/H_{i+1} is generated by an idempotent, and in the second case it is nilpotent.

(4.2.5) LEMMA: The number of factors which are generated by an idempotent is equal to 2^n, where $n=|R|$.

Proof: The factors which are generated by idempotents correspond to elements $w \in W$ such that $a_w^2 = (-1)^{l(w)} a_w$. Let $w \in W$ be such an element. Write $w = w_{i_1} \ldots w_{i_s}$, where $l(w) = s$, and let $J = \{w_{i_j} : 1 \leq j \leq s\}$. Then $w \in W_J$, and by (4.1.2(3)), $w(\Pi_J) \subseteq \Phi^-$. Hence $w = w_{oJ}$. Conversely, for each subset J of R, $a_{w_{oJ}}^2 = (-1)^{l(w_{oJ})} a_{w_{oJ}}$. Hence the number of factors which are generated by an idempotent is equal to the number of subsets of R, i.e. 2^n, where $n = |R|$.

By Schreier's theorem, any series of ideals of H can be refined to a composition series, and all so obtained have the same number of terms in them as the natural series, and with the factors in one-one correspondence with those of the natural series. In particular, consider

$$H > N > 0.$$

This can be refined to a composition series

$$H = H_1' > H_2' > .. > H_r' = N > .. > H'_{|W|} > 0.$$

Now each factor H_i'/H_{i+1}', $i \geqslant r$, is nilpotent as $H_i' \leqslant N$, and each factor H_i'/H_{i+1}', $i+1 \leqslant r$, must be generated by an idempotent as $H_i'/N \leqslant H/N$, a semi-simple ring. Hence the number of factors which are nilpotent is equal to the dimension of N. Thus, $\dim N = |W| - 2^n$, where $n = |R|$. We can, in fact, give a precise basis of N.

(4.2.6) <u>THEOREM</u>: Let $w \in W$, and suppose $w \neq w_{oJ}$ for any $J \subseteq R$. Write $w = w_{i_1} .. w_{i_s}$, $l(w)=s$, and let $J=J(w)=\{w_{i_j} : 1 \leqslant j \leqslant s\}$. Then $E(w) = a_w + (-1)^{l(w_{oJ(w)})+l(w)+1} a_{w_{oJ(w)}}$ is nilpotent, and $\{E(w) : w \in W,\ w \neq w_{oJ} \text{ for any } J \subseteq R\}$ is a basis of N.

<u>Proof</u>: We show that E(w) is nilpotent by induction on $l(w_{oJ(w)}) - l(w)$. Note that if $w = w_{oJ}$ for some $J \subseteq R$, then $E(w_{oJ}) = 0$. Suppose $l(w_{oJ(w)}) - l(w) = 1$. Then since a reduced expression for w involves all $w_i \in J(w)$, $w \neq w_{oJ(w)}$, there exists $r_j \in \Pi_{J(w)}$ such that $w(r_j) \in \Phi^+$. So $a_w^{\,2} = (-1)^{l(w)-1} a_{w_{oJ(w)}}$. Thus,

$$E(w)^2 = a_w^{\,2} + a_w a_{w_{oJ(w)}} + a_{w_{oJ(w)}} a_w + a_{w_{oJ(w)}}^{\,2}$$

So, $E(w)^2 = a_{w_{oJ(w)}}\left((-1)^{l(w)-1} + 2(-1)^{l(w)} + (-1)^{l(w_{oJ(w)})}\right)$

$= 0$, as $l(w_{oJ(w)}) = l(w)+1$.

Now suppose $l(w_{oJ(w)})-l(w) > 1$. Consider the product $a_w a_w$. Since $w \neq w_{oJ(w)}$, there exists $r_i \in \Pi_{J(w)}$ such that $w(r_i) \in \phi^+$. As any reduced expression for w involves all $w_i \in J(w)$, we have $a_w a_w = (-1)^{2l(w)-l(w')} a_{w'}$, with $w' \in W_{J(w)}$ and $l(w') > l(w)$. Further, $J(w') = J(w)$.

Then $E(w)^2 = a_w{}^2 + 2(-1)^{l(w_{oJ(w)})+1} a_{w_{oJ(w)}}$

$$+ (-1)^{l(w_{oJ(w)})} a_{w_{oJ(w)}}$$

$$= (-1)^{l(w')} a_{w'} + (-1)^{l(w_{oJ(w)})+1} a_{w_{oJ(w)}}$$

$$= (-1)^{l(w')}\left(a_{w'} + (-1)^{l(w_{oJ(w')})+l(w')+1} a_{w_{oJ(w')}}\right)$$

as $J(w)=J(w')$

$$= (-1)^{l(w')} E(w').$$

As $l(w') > l(w)$, we have either $w'=w_{oJ(w)}$, and so $E(w')=0$, or $w' \neq w_{oJ(w)}$, and then by induction on $l(w_{oJ(w)})-l(w)$, we have that $E(w')$ is nilpotent. So $E(w)$ is nilpotent.

Finally note that we get a nilpotent element for each $w \in W$, $w \neq w_{oJ}$ for any $J \subseteq R$. The set of all $E(w)$, $w \neq w_{oJ}$ for any $J \subseteq R$, is obviously linearly independent, and there are $|W|-2^n$ elements in all, where $n=|R|$. Hence they are a K-basis for N.

(4.2.7) COROLLARY: ${}^{H}/N$ is commutative.

Proof: We show that $a_{w_i}a_{w_j}-a_{w_j}a_{w_i} \in N$ for all $w_i, w_j \in R$.

If $a_{w_i}a_{w_j}=a_{w_j}a_{w_i}$ the result is obvious. So suppose $a_{w_i}a_{w_j} \neq a_{w_j}a_{w_i}$. Then we can form $E(w_iw_j)$ and $E(w_jw_i)$, and $E(w_iw_j)-E(w_jw_i) = a_{w_i}a_{w_j}-a_{w_j}a_{w_i} \in N$ as each of $E(w_iw_j)$ and $E(w_jw_i)$ is.

We now give some examples of this basis of N.

(1) H_K of type $(W(A_2), \{w_1, w_2\})$

N has dimension 2 and K-basis $\{a_{w_1w_2}+a_{w_0}, a_{w_2}a_{w_1}+a_{w_0}\}$, where $w_0=w_1w_2w_1$.

(2) H_K of type $(W(A_3), \{w_1, w_2, w_3\})$

N has dimension 16, and K-basis:

$a_{w_1w_2}+a_{w_1w_2w_1}$ | $a_{w_1w_2w_3w_2}-a_{w_0}$
$a_{w_2w_1}+a_{w_1w_2w_1}$ | $a_{w_1w_2w_1w_3}-a_{w_0}$
$a_{w_2w_3}+a_{w_2w_3w_2}$ | $a_{w_3w_2w_1w_3}-a_{w_0}$
$a_{w_3w_2}+a_{w_2w_3w_2}$ | $a_{w_3w_1w_2w_1}-a_{w_0}$
$a_{w_3w_2w_1}+a_{w_0}$ | $a_{w_2w_1w_3w_2}-a_{w_0}$
$a_{w_1w_3w_2}+a_{w_0}$ | $a_{w_1w_2w_1w_3w_2}+a_{w_0}$
$a_{w_1w_2w_3}+a_{w_0}$ | $a_{w_1w_3w_2w_1w_3}+a_{w_0}$
$a_{w_2w_1w_3}+a_{w_0}$ | $a_{w_2w_3w_1w_2w_1}+a_{w_0}$

where $w_0 = w_1w_2w_1w_3w_2w_1$.

(4.3) The Irreducible Representations of H.

We investigate the one-dimensional H-modules which arise from the natural composition series of H. Let the factor H_i/H_{i+1} be generated as left H-module by $a_w + H_{i+1}$. The action of H on this element is determined as follows: for each $w_i \in R$,

$$a_{w_i}(a_w + H_{i+1}) = \begin{cases} -(a_w + H_{i+1}) & \text{if } w^{-1}(r_i) \in \phi^- \\ 0 & \text{if } w^{-1}(r_i) \in \phi^+ \end{cases}$$

For any $w \in W$, let $J(w)=\{w_{i_j} : 1 \leq j \leq s\}$ where $w = w_{i_1} \cdots w_{i_s}$ is a reduced expression for w. Then

$$a_{w'}(a_w + H_{i+1}) = \begin{cases} (-1)^{l(w')}(a_w + H_{i+1}) \\ \qquad \text{if } w^{-1}(\prod_{J(w')}) \subseteq \phi^- \\ 0 \text{ if there exists } r_i \in \prod_{J(w')} \\ \qquad \text{such that } w^{-1}(r_i) \in \phi^+. \end{cases}$$

Hence the action of H on $a_w + H_{i+1}$ depends on the element $w^{-1} \in W$.

(4.3.1) **Definition**: For each $J \subseteq R$, let λ_J be the one-dimensional representation of H defined by

$$\lambda_J(a_{w_i}) = \begin{cases} 0 & \text{if } w_i \in J \\ -1 & \text{if } w_i \notin J \end{cases}$$

For all $w \in W$, let $w = w_{i_1} \cdots w_{i_s}$ with $l(w) = s$. Then $\lambda_J(a_w) = \lambda_J(a_{w_{i_1}}) \cdots \lambda_J(a_{w_{i_s}})$. Extend λ_J by linearity to H.

For each $J \subseteq R$, let $H_{i(J)}/H_{i(J)+1}$ be the factor of the natural series of H which is generated by

$a_{w_0\hat{J}} + H_{i(J)+1}$. Then the left H-module $H_{i(J)}/H_{i(J)+1}$ affords the representation λ_J of H.

Since each composition factor of H is one-dimensional, it follows that all irreducible representations of H are one-dimensional. Let μ be an irreducible representation of H. Then μ is completely determined by the values $\mu(a_{w_i})$ for all $w_i \in R$. Since μ is an algebra homomorphism, we must have that $\mu(a_{w_i})^2 = -\mu(a_{w_i})$ for all $w_i \in R$. Let $\mu(a_{w_i})=u_i \in K$ for all $w_i \in R$. Then $u_i^2 = -u_i$ in K implies that $u_i=0$ or $u_i = -1$. Thus we can describe each irreducible representation of H by an n-tuple $(u_1,..u_n)$, where $n=|R|$, with $u_i=0$ or -1 for all i. In particular, λ_J corresponds to the n-tuple $(u_1,..u_n)$ where $u_i=0$ if $w_i \in J$ and $u_i=-1$ if $w_i \notin J$. There are 2^n such irreducible representations, and they all occur in the natural series of H.

We determine 2^n maximal left ideals of H as follows: for each $J \subseteq R$, form the n-tuple $(u_1,..u_n)$, where $u_i=0$ if $w_i \in J$ and is -1 otherwise. Let M_J be the left ideal of H generated by $\{a_{w_i}-u_i1 : w_i \in R\}$. Then $M_J = \ker \lambda_J$, and as λ_J is irreducible, M_J is a maximal left ideal of H.

Now, H/N is semi-simple Artinian, so if we extend K to its algebraic closure $\bar{K}$ and consider H as an algebra over $\bar{K}$, we deduce that

$^H/_N \cong \bar{K} \oplus \bar{K} \oplus .. \oplus \bar{K}$, a direct sum of 2^n fields.

(Actually, we will show that

$$^H/_N \cong K \oplus K \oplus .. \oplus K,\ 2^n \text{ copies of } K,$$

regardless of which field K is.)

(4.4) Some Decompositions of H.

In this section we will determine two decompositions of H as a direct sum of 2^n left ideals.

For each subset J of R, let H_J be the subalgebra of H generated by $\{a_{w_i} : w_i \in J\}$. Define elements e_J and o_J in H_J as follows:

$$(4.4.1) \qquad \begin{aligned} e_J &= \sum_{w \in W_J} a_w \\ o_J &= (-1)^{l(w_{oJ})} a_{w_{oJ}} \end{aligned}$$

e_J and o_J are in the centre of H_J by inspection.

(4.4.2) LEMMA: Let $w_{oJ} = w_{i_1} .. w_{i_s}$, $l(w_{oJ})=s$. Then

$$e_J = (1 + a_{w_{i_1}})(1 + a_{w_{i_2}})..(1 + a_{w_{i_s}})$$

and is independent of the reduced expression for w_{oJ}.

Notation: For all $w \in W$, if $w = w_{i_1} .. w_{i_t}$ with $l(w)=t$, write $\left[1 + a_w\right] = (1 + a_{w_{i_1}})..(1 + a_{w_{i_t}})$. By the following proof, it follows that $\left[1 + a_w\right]$ is independent of the reduced expression for w.

Proof: Firstly we show that $\left[1 + a_{w_{oJ}}\right]$ is independent of the reduced expression for w_{oJ}. Since we can pass from one reduced expression for w_{oJ} to another by substitutions of the form $(w_iw_j\cdot\cdot)_{n_{ij}} = (w_jw_i\cdot\cdot)_{n_{ij}}$, $i\neq j$, where n_{ij} is the order of w_iw_j in W, we need to show that

$$\left[1 + a_{(w_iw_j\cdot\cdot)_{n_{ij}}}\right] = \left[1 + a_{(w_jw_i\cdot\cdot)_{n_{ij}}}\right].$$

To do this, we use induction on n, $n\leq n_{ij}$, to show that

$$\left[1 + a_{(w_iw_j\cdot\cdot)_n}\right] = 1 + \sum_{m=1}^{n} a_{(w_iw_j\cdot\cdot)_m} + \sum_{m=1}^{n-1} a_{(w_jw_i\cdot\cdot)_m}.$$

This is clearly true for n=1. Suppose it is true for all integers $\leq k$, and suppose that k is odd. Then

$$\begin{aligned}\left[1 + a_{(w_iw_j\cdot\cdot)_{k+1}}\right] &= \left[1 + a_{(w_iw_j\cdot\cdot)_k}\right](1 + a_{w_j}) \\ &= (1 + \sum_{m=1}^{k} a_{(w_iw_j\cdot\cdot)_m} + \sum_{m=1}^{k-1} a_{(w_jw_i\cdot\cdot)_m})(1 + a_{w_j}) \\ &= (1 + \sum_{m=1}^{k} a_{(w_iw_j\cdot\cdot)_m} + \sum_{m=1}^{k-1} a_{(w_jw_i\cdot\cdot)_m}) + a_{w_j} \\ &\quad + \sum_{m=0}^{\frac{1}{2}(k-1)} a_{(w_iw_j\cdot\cdot)_{2m+1}}a_{w_j} + \sum_{m=1}^{\frac{1}{2}(k-1)} a_{(w_iw_j\cdot\cdot)_{2m}}a_{w_j} \\ &\quad + \sum_{m=1}^{\frac{1}{2}(k-1)} a_{(w_jw_i\cdot\cdot)_{2m-1}}a_{w_j} + \sum_{m=1}^{\frac{1}{2}(k-1)} a_{(w_jw_i\cdot\cdot)_{2m}}a_{w_j}.\end{aligned}$$

Now, $a_{(w_iw_j\cdot\cdot)_{2m-1}}a_{w_j} = -a_{(w_iw_j\cdot\cdot)_{2m}}a_{w_j}$, $1\leq m\leq\frac{1}{2}(k-1)$, and

$a_{(w_jw_i\cdot\cdot)_{2m-1}}a_{w_j} = -a_{(w_jw_i\cdot\cdot)_{2m-2}}a_{w_j}$, $1\leq m\leq\frac{1}{2}(k-1)$, where

$a_{(w_jw_i\cdot\cdot)_0}=1$. Then

$$\begin{aligned}\left[1 + a_{(w_iw_j\cdot\cdot)_{k+1}}\right] &= 1 + \sum_{m=1}^{k} a_{(w_iw_j\cdot\cdot)_m} + \sum_{m=1}^{k-1} a_{(w_jw_i\cdot\cdot)_m} \\ &\quad + a_{(w_iw_j\cdot\cdot)_k}a_{w_j} + a_{(w_jw_i\cdot\cdot)_{k-1}}a_{w_j}\end{aligned}$$

i.e. $\left[1 + a_{(w_i w_j\cdot\cdot)_{k+1}}\right] = 1 + \sum_{m=1}^{k+1} a_{(w_i w_j\cdot\cdot)_m} + \sum_{m=1}^{k} a_{(w_j w_i\cdot\cdot)_m}$

Similarly, we get the same result if we had assumed that k was even.

Similarly, we can show that for all n, $n \leq n_{ij}$,

$$\left[1 + a_{(w_j w_i\cdot\cdot)_n}\right] = 1 + \sum_{m=1}^{n} a_{(w_j w_i\cdot\cdot)_m} + \sum_{m=1}^{n-1} a_{(w_i w_j\cdot\cdot)_m}$$

So for all $n \leq n_{ij}$, $\left[1 + a_{(w_i w_j\cdot\cdot)_n}\right] - \left[1 + a_{(w_j w_i\cdot\cdot)_n}\right]$

$$= a_{(w_i w_j\cdot\cdot)_n} - a_{(w_j w_i\cdot\cdot)_n}.$$

When $n=n_{ij}$, this difference is zero. So

$$\left[1 + a_{(w_i w_j\cdot\cdot)_{n_{ij}}}\right] = \left[1 + a_{(w_j w_i\cdot\cdot)_{n_{ij}}}\right]$$

and thus $\left[1 + a_{w_{oJ}}\right]$ is independent of the reduced expression for w_{oJ} chosen.

Finally, $\left[1 + a_{w_{oJ}}\right]$ is a linear combination of certain a_w with $w \in W_J$. We show by induction on $l(w)$ for all $w \in W_J$ that a_w occurs in the expansion of $\left[1 + a_{w_{oJ}}\right]$ with coefficient 1. If $l(w)=0$, then $w=1$ and obviously $a_1=1$ occurs with coefficient 1. Suppose $l(w) > 0$. Let $w = w'w_j$, $w' \in W_J$, $w_j \in J$, where $l(w) = l(w')+1$. By induction $a_{w'}$ occurs in $\left[1 + a_{w_{oJ}}\right]$ with coefficient 1. Choose an expression for w_{oJ} ending in w_j, and then

$$\left[1 + a_{w_{oJ}}\right] = \left[1 + a_{w_{oJ}w_j}\right] (1 + a_{w_j})$$

Since $l(w'w_j) > l(w')$, the only contribution to $a_{w'}$ from the last bracket is from the 1. If instead we take a_{w_j} from the last bracket, we get a_w, with coefficient 1. Now suppose a_w occurs in $\left[1 + a_{w_{oJ}w_j}\right]$ with coefficient m. Then

$$\begin{aligned} ma_w(1 + a_{w_j}) &= ma_w + ma_w a_{w_j} \\ &= ma_w - ma_w \text{ as } w(r_j) \in \phi^- \\ &= 0. \end{aligned}$$

Thus a_w occurs in the expansion of $\left[1 + a_{w_{oJ}}\right]$ with coefficient 1, and hence $e_J = \left[1 + a_{w_{oJ}}\right]$.

(4.4.3) <u>COROLLARY</u>: e_J and o_J are idempotents in the centre of H_J.

<u>Proof</u>: For all $w_i \in R$, $(1 + a_{w_i})(1 + a_{w_i}) = (1 + a_{w_i})$, and $a_{w_i}{}^2 = -a_{w_i}$. The result follows as $e_J = \left[1 + a_{w_{oJ}}\right]$, and w_{oJ} has a reduced expression ending in w_j for all $w_j \in J$.

(4.4.4) <u>LEMMA</u>: (1) If $J, L \subseteq R$, $o_J e_L = 0$ if $J \cap L \neq \emptyset$ and $e_J o_L = 0$ if $J \cap L \neq \emptyset$.

(2) If $L \subseteq J \subseteq R$, $e_L e_J = e_J = e_J e_L$ and $o_L o_J = o_J = o_J o_L$.

<u>Proof</u>: (1) If $J \cap L \neq \emptyset$, choose $w_i \in J \cap L$. Then we can choose reduced expressions for w_{oJ} and w_{oL} ending and beginning respectively with w_i. Then $o_J e_L = ..a_{w_i}(1+a_{w_i})...$ and so $o_J e_L = 0$. Similarly, $e_J o_L = ..(1+a_{w_i})a_{w_i}.. = 0$.

(2) For each $w_i \in J$, $(1+a_{w_i})e_J = e_J + a_{w_i}e_J$. Since $a_{w_i} = -o_{\{w_i\}}$, $(1+a_{w_i})e_J = e_J$ by (1). Also, $a_{w_i}o_J = -o_J$ if $w_i \in J$, and thus the results follow.

(4.4.5) <u>LEMMA</u>: Let $y \in Y_J$ for some $J \subseteq R$. Then $a_y o_{\hat{J}} = a_y$ and $a_y o_{\hat{J}} e_J = \sum_{w \in W_J} a_{yw}$, with $l(yw) = l(y)+l(w)$ for all $w \in W_J$.

<u>Proof</u>: By (1.3.4), if $y \in Y_J$, then $y = ww_{o\hat{J}}$ for some $w \in W$ with $l(y) = l(w)+l(w_{o\hat{J}})$. Hence $a_y o_{\hat{J}} = (-1)^{l(w_{o\hat{J}})} a_w a_{w_{o\hat{J}}} a_{w_{o\hat{J}}}$, and so $a_y o_{\hat{J}} = a_y$. Now for all $w \in W_J$, as $y \in Y_J \subseteq X_J$, by (1.3.2) we have $l(yw) = l(y)+l(w)$. So for all $w \in W_J$ $a_y a_w = a_{yw}$. Thus $a_y o_{\hat{J}} e_J = a_y e_J = \sum_{w \in W_J} a_y a_w = \sum_{w \in W_J} a_{yw}$, and $l(yw) = l(y)+l(w)$ for all $w \in W_J$.

(4.4.6) <u>LEMMA</u>: For $y \in Y_J$, a_y occurs in the expansion of $a_y e_J o_{\hat{J}}$ with coefficient 1, and if, for any $w \in W$, a_w occurs in the expansion of $a_y e_J o_{\hat{J}}$ with non-zero coefficient, then $w=y$ or $l(w) > l(y)$.

<u>Proof</u>: By (4.4.5), $a_y e_J = \sum_{w \in W_J} a_{yw}$ with $l(yw) = l(y)+l(w)$ for all $w \in W_J$. Then

$$a_y e_J o_{\hat{J}} = \sum_{w \in W_J} a_{yw} o_{\hat{J}} = a_y o_{\hat{J}} + \sum_{\substack{w \in W_J \\ w \neq 1}} a_{yw} o_{\hat{J}}.$$

By (4.4.5) $a_y o_{\hat{J}} = a_y$, and for all $w \in W_J$, $w \neq 1$, $a_{yw}(-1)^{l(w_{oJ})} a_{w_{oJ}} = \pm a_{w'}$ for some $w' \in W$, with $l(w') \geqslant l(yw) > l(y)$.

(4.4.7) <u>THEOREM</u>: (1) The elements $\{a_y o_{\hat{J}} e_J : y \in Y_J, J \subseteq R\}$ are linearly independent and form a basis of H.

(2) The elements $\{a_y e_J o_{\hat{J}} : y \in Y_J, J \subseteq R\}$ are linearly independent and form a basis of H.

Proof: (1) Suppose that for each $y \in Y_J$ and each $J \subseteq R$ there is an element $k_y \in K$ such that $\sum_{J \subseteq R} \sum_{y \in Y_J} k_y a_y o_{\hat{J}} e_J = 0$.

Let $S_n = \sum_{J \subseteq R} \sum_{\substack{y \in Y_J \\ l(y) \geq n}} k_y a_y o_{\hat{J}} e_J$. We show that if $S_n = 0$, then $k_y = 0$ whenever $l(y)=n$ and hence $S_{n+1} = 0$.

Let $y_1, \ldots y_t$ be those elements of W for which $l(y_i)=n$. Then by (4.4.5), if $y_i \in Y_{J(i)}$ for some $J(i) \subseteq R$,

$$a_{y_i} o_{\widehat{J(i)}} e_{J(i)} = a_{y_i} + (\text{a linear combination of certain } a_w \text{ where } l(w) > l(y_i))$$

Hence, $S_n = \sum_{i=1}^{t} k_{y_i} a_{y_i}$ + (a linear combination of certain a_w with $l(w) > n$).

If $S_n = 0$, then as $\{a_w : w \in W\}$ are a basis of H, we must have $k_{y_i} = 0$ for all i, $1 \leq i \leq t$. So $S_{n+1} = 0$.

Since $S_0 = 0$, $k_y = 0$ for all y whenever $l(y) = 0$, and $S_1 = 0$. By induction, all k_y are zero, and so $\{a_y o_{\hat{J}} e_J : y \in Y_J, J \subseteq R\}$ is a set of linearly independent elements. As there are $|W|$ of them, as $\sum_{J \subseteq R} |Y_J| = |W|$, they must form a basis of H.

(2) A similar argument gives the result, since by (4.4.6) $a_y e_J o_{\hat{J}} = a_y$ + (a linear combination of certain a_w where $l(w) > l(y)$) for $y \in Y_J$.

(4.4.8) <u>COROLLARY</u>: (1) For any $L \subseteq R$, the elements of the set $\{a_y o_{\hat{J}} e_J o_{\hat{L}} : y \in Y_J,\ J \subseteq L\}$ are linearly independent.

(2) For any $L \subseteq R$, the elements of the set $\{a_y e_J o_{\hat{J}} e_L : y \in Y_J,\ L \subseteq J\}$ are linearly independent.

<u>Proof</u>: (1) For any $y \in Y_J$, $a_y o_{\hat{J}} = a_y$, and so $a_y o_{\hat{J}} e_J o_{\hat{L}} = a_y e_J o_{\hat{L}}$. Then $a_y e_J o_{\hat{L}} = \sum_{w \in W_J} a_{yw} o_{\hat{L}}$. As $J \subseteq L$, $\hat{L} \subseteq \hat{J}$ and so $a_{w_{o\hat{J}}} o_{\hat{L}} = a_{w_{o\hat{J}}}$

$$\text{Then } a_y e_J o_{\hat{L}} = a_y o_{\hat{L}} + \sum_{\substack{w \in W_J \\ w \neq 1}} a_{yw} o_{\hat{L}} = a_y + \sum_{\substack{w \in W_J \\ w \neq 1}} a_{yw} o_{\hat{L}} \text{ as } y \in Y_J$$

$$= a_y + (\text{a linear combination of certain } a_w \text{ with } l(w) > l(y))$$

Now we may use a similar argument to that used in the proof of (4.4.7) to deduce that the given elements are linearly independent.

(2) For any $y \in Y_J$, $a_y e_J o_{\hat{J}} = a_y + (\sum_{w \in W} k_w a_w)$ where $k_w \in K$ and $k_w = 0$ if $l(w) \leq l(y)$. Then

$$a_y e_J o_{\hat{J}} e_L = a_y e_L + (\sum_{w \in W} k_w a_w) e_L,\ k_w \in K \text{ given as above,}$$
$$= a_y + (\sum_{w \in W} k'_w a_w) \text{ for certain } k'_w \in K, \text{ with } k'_w = 0 \text{ if } l(w) \leq l(y).$$

Once again we can use a similar argument to that in the proof of (4.4.7) to get the result.

(4.4.9) <u>THEOREM</u>: (1) For each $a \in H$, and for any $J \subseteq R$, there exist elements $k_y \in K$ such that

$$a o_{\hat{J}} e_J = \sum_{y \in Y_J} k_y a_y o_{\hat{J}} e_J .$$

(2) For each $a \in H$, and for any $J \subseteq R$, there exist elements $k_y \in K$ such that

$$ae_J o_{\hat{J}} = \sum_{y \in Y_J} k_y a_y e_J o_{\hat{J}}.$$

Proof: We will prove the first part by a method which cannot be directly applied to the second part. However, the method we will use to prove the second part can be altered slightly to apply to the first.

(1) As $\{a_w : w \in W\}$ is a basis of H, we may write $a = \sum_{w \in W} u_w a_w$, with $u_w \in K$ for all $w \in W$. It is thus sufficient to express $a_w o_{\hat{J}} e_J$ as a linear combination of the elements $\{a_y o_{\hat{J}} e_J : y \in Y_J\}$ for all $w \in W$. Use induction on $l(w)$ to prove this.

If $l(w)=0$, then $w=1$, and $o_{\hat{J}} e_J = (-1)^{l(w_{o\hat{J}})} a_{w_{o\hat{J}}} o_{\hat{J}} e_J$ as $o_{\hat{J}} = o_{\hat{J}}^2 = (-1)^{l(w_{o\hat{J}})} a_{w_{o\hat{J}}} o_{\hat{J}}$. Result is true for $w=1$ as $a_{w_{o\hat{J}}} \in Y_J$.

Suppose $l(w) > 0$. Let $w = w_i w'$ for some $w_i \in R$, $w' \in W$, $l(w) = l(w')+1$. By induction,

$$a_{w'} o_{\hat{J}} e_J = \sum_{y \in Y_J} u_y a_y o_{\hat{J}} e_J \text{ for some } u_y \in K.$$

Then $a_w o_{\hat{J}} e_J = a_{w_i} a_{w'} o_{\hat{J}} e_J = \sum_{y \in Y_J} u_y a_{w_i} a_y o_{\hat{J}} e_J$. Hence for each $y \in Y_J$ we have to express $a_{w_i} a_y o_{\hat{J}} e_J$ as a combination of $\{a_v o_{\hat{J}} e_J : v \in Y_J\}$. Now for any $y \in Y_J$,

$$(4.4.10)\quad a_{w_i}a_y o_{\hat J} e_J = \begin{cases} -a_y o_{\hat J} e_J \text{ if } y^{-1}(r_i) \in \phi^- \\ 0 \text{ if } y^{-1}(r_i)=r_j \text{ for some } r_j \in \prod_J, \\ \qquad \text{as then } a_{w_i} a_y o_{\hat J} = a_y o_{\hat J} a_{w_j} \\ a_{w_i y} o_{\hat J} e_J \text{ where } w_i y \in Y_J \text{ if } y^{-1}(r_i) \in \phi^+, \\ \qquad y^{-1}(r_i) \neq r_j \text{ for any } r_j \in \prod_J. \end{cases}$$

Hence the result follows.

(2) Since $\{a_y e_L o_{\hat L} : y \in Y_L,\ L \subseteq R\}$ is a basis of H, there exist elements $u_y \in K$ such that

$$ae_J o_{\hat J} = \sum_{L \subseteq R} \sum_{y \in Y_L} u_y a_y e_L o_{\hat L}.$$

Choose any $M \subseteq R$ with $M \cap \hat J \neq \emptyset$. Then $ae_J o_{\hat J} e_M = 0$, so

$$\sum_{L \subseteq R} \sum_{y \in Y_L} u_y a_y e_L o_{\hat L} e_M = 0.$$

But $o_{\hat L} e_M = 0$ if $\hat L \cap M \neq \emptyset$. So the only non-zero terms in the above equation involve those $L \subseteq R$ for which $\hat L \cap M = \emptyset$.

Thus $\sum\limits_{\substack{L \\ M \subseteq L \subseteq R}} \sum\limits_{y \in Y_L} u_y a_y e_L o_{\hat L} e_M = 0$. By (4.4.8(2)), $u_y=0$

for all $y \in Y_L$, $M \subseteq L \subseteq R$. Hence we have that $u_y=0$ for all $y \in Y_L$, with $L \cap \hat J \neq \emptyset$. So we now have

$$ae_J o_{\hat J} = \sum_{\substack{L \\ L \subseteq J}} \sum_{y \in Y_L} u_y a_y e_L o_{\hat L}.$$

Let $S_J = \{w \in W : u_w \neq 0,\ w \in Y_L \text{ for some } L \subset J\}$. Suppose $S_J \neq \emptyset$. Choose an element $y_o \in S_J$ of minimal length, and suppose $y_o \in Y_{J_o}$ for some $J_o \subset J$. Consider

$$ae_J o_{\hat J} o_{\hat J_o} = \sum_{\substack{L \\ L \subseteq J}} \sum_{y \in Y_L} u_y a_y e_L o_{\hat L} o_{\hat J_o}.$$

As $J_o \subset J$, $ae_J o \hat{J} o \hat{J}_o = 0$. Then

$$\sum_{L \subset J} \sum_{y \in Y_L} u_y a_y e_L o \hat{L} o \hat{J}_o = 0 \qquad (*)$$

Now if $L \subset J$ and $y \in Y_L$,

$$a_y e_L o \hat{L} o \hat{J}_o = a_y o \hat{J}_o + \sum_{\substack{w \in W \\ l(w) > l(y)}} k_w a_w \quad \text{where } k_w \in K,$$

and $a_y o \hat{J}_o = \pm a_w$, for some $w \in W$ with $l(w) \geqslant l(y)$.

Since y_o is of minimal length in S_J, the coefficient of a_{y_o} on the left side of (*) is u_{y_o}. As $\{a_w : w \in W\}$ is a basis of H, so $u_{y_o} = 0$ - contradiction. Hence $S_J = \emptyset$ and $ae_J o \hat{J} = \sum_{y \in Y_J} u_y a_y e_J o \hat{J}$.

Remark: Let $z \in Z$. We can regard z as an element of K in a natural way - it is the element $z1_K = 1_K + 1_K + .. + 1_K$ (z times) where 1_K is the identity of K.

(4.4.11) COROLLARY: (1) For each $w \in W$, there exist rational integers $u_y = u_y(w)$ such that $a_w o \hat{J} e_J = \sum_{y \in Y_J} u_y a_y o \hat{J} e_J$.

(2) For each $w \in W$, there exist rational integers $u_y = u_y(w)$ such that $a_w e_J o \hat{J} = \sum_{y \in Y_J} u_y a_y e_J o \hat{J}$.

Proof: (1) Follows from the proof of (4.4.9(1)).

(2) List the elements $y_1, .. y_m$ of Y_J in order of increasing length; if $i < j$ then $l(y_i) \leqslant l(y_j)$. Let c_{ij} be the coefficient of a_{y_i} in $a_{y_j} e_J o \hat{J}$. Clearly c_{ij} is an integer as $a_{y_j} e_J o \hat{J}$ is an integral combination of certain elements $a_{w'}$, $w' \in W$. Also, $c_{ii} = 1$ for all i, $1 \leqslant i \leqslant m$, and $c_{ij} = 0$

if $i < j$ by (4.4.6). Let h_i be the coefficient of a_{y_i} in $a_w e_J o_{\hat{J}}$. Clearly h_i is an integer, and

$$h_i = \sum_{j=1}^{m} k_j c_{ij} \quad \text{where } a_w e_J o_{\hat{J}} = \sum_{i=1}^{m} k_i a_{y_i} e_J o_{\hat{J}} \quad \text{for}$$

some $k_i \in K$. Hence,

$$h_i = \sum_{j=1}^{i-1} k_j c_{ij} + k_i$$

Let $i=1$. Then $h_1 = k_1$, an integer. Now use increasing induction on i to show k_i is an integer for all i, $1 \leq i \leq m$.

(4.4.12) THEOREM: (1) $Ho_{\hat{J}}e_J$ is a left ideal of H with K-basis $\{a_y o_{\hat{J}} e_J : y \in Y_J\}$. Hence $\dim Ho_{\hat{J}}e_J = |Y_J|$. Moreover, $H = \sum^{\oplus}_{J \subseteq R} Ho_{\hat{J}}e_J$, a direct sum of 2^n left ideals, where $n=|R|$.

(2) $He_J o_{\hat{J}}$ is a left ideal of H with K-basis $\{a_y e_J o_{\hat{J}} : y \in Y_J\}$. Hence $\dim He_J o_{\hat{J}} = |Y_J|$. Moreover, $H = \sum^{\oplus}_{J \subseteq R} He_J o_{\hat{J}}$, a direct sum of 2^n left ideals, where $n=|R|$.

Proof: The results follow by (4.4.7), (4.4.9) and the fact that $\dim H = |W| = \sum_{J \subseteq R} |Y_J|$.

(4.4.13) COROLLARY: $Ho_{\hat{J}}e_J$ and $He_J o_{\hat{J}}$ are indecomposable left ideals of H, for all $J \subseteq R$, and they are isomorphic as left ideals of H.

Proof: From the theory of Artinian rings and the fact that H/N is a direct sum of 2^n irreducible components (see remarks at the end of (4.3)), it follows that H can be expressed as the direct sum of 2^n indecomposable left ideals. Hence we must have that $Ho_{\hat{J}}e_J$ and $He_Jo_{\hat{J}}$ are indecomposable left ideals of H for all $J \subseteq R$.

To show they are isomorphic, we first show that $He_Jo_{\hat{J}} = Ho_{\hat{J}}e_Jo_{\hat{J}}$. Since each element of $He_Jo_{\hat{J}}$ is of the form $\sum_{y \in Y_J} k_y a_y e_J o_{\hat{J}}$ for some $k_y \in K$, and each $y \in Y_J$ is of the form $y = ww_{o\hat{J}}$ for some $w \in W$ with $l(y) = l(w)+l(w_{o\hat{J}})$, we have that $a_y = a_y(-1)^{l(w_{o\hat{J}})} a_{w_{o\hat{J}}} = a_y o_{\hat{J}}$. Thus $\sum_{y \in Y_J} k_y a_y e_J o_{\hat{J}} = \sum_{y \in Y_J} k_y a_y o_{\hat{J}} e_J o_{\hat{J}}$ and so $He_Jo_{\hat{J}} \leqslant Ho_{\hat{J}}e_Jo_{\hat{J}}$. But obviously $Ho_{\hat{J}}e_Jo_{\hat{J}} \leqslant He_Jo_{\hat{J}}$, and so we have equality.

Now define the homomorphism f_J of left ideals of H

f_J: $Ho_{\hat{J}}e_J \to He_Jo_{\hat{J}}$, by $f_J(ao_{\hat{J}}e_J) = ao_{\hat{J}}e_Jo_{\hat{J}}$, for all $ao_{\hat{J}}e_J \in Ho_{\hat{J}}e_J$. As f_J is given by right multiplication by $o_{\hat{J}}$, it is well-defined and is a homomorphism of left ideals of H. f_J is onto, since $He_Jo_{\hat{J}} = Ho_{\hat{J}}e_Jo_{\hat{J}}$ and an element $ao_{\hat{J}}e_Jo_{\hat{J}} \in He_Jo_{\hat{J}}$ is the image under f_J of $ao_{\hat{J}}e_J$. f_J is one-one as $\dim Ho_{\hat{J}}e_J = \dim He_Jo_{\hat{J}}$. Hence f_J is an isomorphism of left ideals of H.

(4.4.14) <u>COROLLARY</u>: (1) For any $L \subseteq R$,

$$Ho_{\hat{L}} = \sum_{\substack{J \\ J \subseteq L}}^{\oplus} Ho_{\hat{J}}e_Jo_{\hat{L}}, \quad \dim Ho_{\hat{L}} = \sum_{J \subseteq L} |Y_J| = |X_{\hat{L}}|.$$

(2) For any $L \subseteq R$,

$$He_L = \sum_{\substack{J \\ L \subseteq J}}^{\oplus} He_Jo_{\hat{J}}e_L, \quad \dim He_L = \sum_{L \subseteq J} |Y_J| = |X_L|.$$

Proof: Use (4.4.12) and (4.4.8).

(4.4.15) THEOREM: For any $J \subseteq R$,

$$He_J = \{a \in H : aa_{w_i} = 0 \text{ for all } w_i \in J\}$$
$$= \{a \in H : a(1 + a_{w_i}) = a \text{ for all } w_i \in J\}.$$

Further, $He_J = \sum^{\oplus}_{J \subseteq L} Ho^{\wedge}_L e_L$, and He_J has basis $\{a_w e_J : w \in X_J\}$ and dimension $|X_J|$. Consider the map $\theta : He_J \to Ho^{\wedge}_J e_J$ given by projection. Then $\theta : {}^{He}J/ \sum_{J \subset L} He_L \cong Ho^{\wedge}_J e_J$. Finally,

$$Ho^{\wedge}_J e_J = \{a \in H : aa_{w_i} = 0 \text{ for all } w_i \in J, ae_L = 0 \text{ for all } L \supset J\}$$

$$= He_J \cap (\bigcap_{J \subset L} \ker e_L), \text{ where}$$

$\ker e_L = \{a \in H : ae_L = 0\}$.

Proof: Clearly $He_J \leq \{a \in H : aa_{w_i} = 0 \text{ for all } w_i \in J\}$. Conversely take $a \in H$ and suppose $aa_{w_i} = 0$ for all $w_i \in J$. Then $a(1 + a_{w_i}) = a$ for all $w_i \in J$, and so $ae_J = a$, and $a \in He_J$. Thus $He_J = \{a \in H : aa_{w_i} = 0 \text{ for all } w_i \in J\}$. Similarly, $He_J = \{a \in H : a(1 + a_{w_i}) = a \text{ for all } w_i \in J\}$.

Then $Ho^{\wedge}_L e_L \leq He_J$ for all $L \supseteq J$, and so $\sum^{\oplus}_{J \subseteq L} Ho^{\wedge}_L e_L \leq He_J$. By (4.4.14), $\dim He_J = |X_J|$, and as $\dim Ho^{\wedge}_L e_L = |Y_L|$, we have $He_J = \sum^{\oplus}_{J \subseteq L} Ho^{\wedge}_L e_L$.

Let $a = \sum_{w \in W} u_w a_w \in He_J$, where $u_w \in K$. Let $w_i \in J$. Then $aa_{w_i} = 0$, and so $\sum_{w \in W} u_w a_w a_{w_i} = 0$.

Now $\sum_{w \in W} u_w a_w a_{w_i} = \sum_{\substack{w \in W \\ w(r_i) \in \phi^+}} u_w a_{ww_i} - \sum_{\substack{w \in W \\ w(r_i) \in \phi^-}} u_w a_w$

That is, $\sum_{\substack{w\in W\\ w(r_i)\in\phi^-}} u_{ww_i}a_w - \sum_{\substack{w\in W\\ w(r_i)\in\phi^-}} u_w a_w = 0.$

Since $\{a_w : w \in W\}$ form a basis of H, we have $u_{ww_i} = u_w$ for all $w \in W$ with $w(r_i) \in \phi^-$. Hence $u_w = u_{ww_i}$ for all $w \in W$, with $w(r_i) \in \phi^+$. Now if $w \in W$, w can be expressed uniquely in the form $w = yw_J$, where $y \in X_J$, $w_J \in W_J$ and $l(w)=l(y)+l(w_J)$. Write $w_J = w_{i_1}..w_{i_t}$, $w_{i_j} \in J$, $l(w_J)=t$. By the above we have $u_y = u_{yw_{i_1}} = u_{yw_{i_1}w_{i_2}} = .. = u_{yw_J} = u_w$.

Hence $a = \sum_{y\in X_J} u_y a_y e_J$. Conversely, for each $y \in X_J$, $a_y e_J \in He_J$, and as $\{a_y e_J : y \in X_J\}$ is linearly independent and $\dim He_J = |X_J|$, $\{a_y e_J : y \in X_J\}$ is a basis of He_J.

Since $He_J = \sum^{\oplus}_{J\subseteq L} Ho_L^{\wedge} e_L$, He_J also has basis $\{a_y o_L^{\wedge} e_L : y \in Y_L,\ L \supseteq J\}$. Consider $\theta : He_J \to Ho_J^{\wedge} e_J$ given by $\theta(\sum_{J\subseteq L}\sum_{w\in Y_L} u_w a_w o_L^{\wedge} e_L) = \sum_{w\in Y_J} u_w a_w o_J^{\wedge} e_J$. Since each $Ho_L^{\wedge} e_L$ is a left H-module, θ is a left H-module homomorphism, and is onto. $\operatorname{Ker}\theta = \{a \in He_J : a = \sum_{\substack{L\\ J\subset L}}\sum_{w\in Y_L} u_w a_w o_L^{\wedge} e_L\}$

$$= \{a \in He_J : a \in \sum^{\oplus}_{\substack{L\\ J\subset L}} Ho_L^{\wedge} e_L\}$$

$$= \sum_{\substack{L\\ J\subset L}} He_L.$$

and $\theta : {}^{He}J/\ker\theta \cong Ho_J^{\wedge} e_J$.

Finally, $Ho_J^{\wedge} e_J \leq \{a \in H : aa_{w_i} = 0$ for all $w_i \in J$, $ae_L = 0$ for all $L \supset J\}$. Let $a = \sum_L \sum_{y\in Y_L} u_y a_y o_L^{\wedge} e_L$, $u_y \in K$, satisfy $aa_{w_i} = 0$ for all $w_i \in J$ and $ae_L = 0$ for all $L \supset J$. Since $a \in He_J$, $u_y = 0$ for all $y \in Y_L$ if $J \not\subseteq L$. So $a = \sum_{J \subseteq L} \sum_{y \in Y_L} u_y a_y o_L^{\wedge} e_L$. Set $S_J = \{w \in W : u_w \neq 0,\ w \in Y_L,\ L \supset J\}$.

Suppose $S_J \neq \emptyset$. There exists an element y_o of minimal length in S_J; suppose $y_o \in Y_M$, $M \supset J$. Then $ae_M = 0$. Also, $o_J^{\wedge} e_J e_M = 0$ as $M \supset J$. For other $L \supset J$, if $y \in Y_L$,

$a_y o_L^{\wedge} e_L e_M = a_y e_L e_M = a_y$ + (a combination of certain a_w, $w \in W$, with $l(w) > l(y)$)

Then $ae_M = 0$ gives $\sum_{J \subset L} \sum_{y \in Y_L} u_y a_y o_L^{\wedge} e_L e_M = 0$. As y_o is of minimal length in S_J, the coefficient of a_{y_o} in the left side of the last equation is u_{y_o}. By the linear independence of $\{a_w : w \in W\}$, we must have $u_{y_o} = 0$ - contradiction. Hence $S_J = \emptyset$ and $a = \sum_{y \in Y_J} u_y a_y o_J^{\wedge} e_J \in Ho_J^{\wedge} e_J$. Thus

$$Ho_J^{\wedge} e_J = \{a \in He_J : ae_L = 0 \text{ for all } L \supset J\}.$$

(4.4.16) <u>THEOREM</u>: For any $J \subseteq R$,

$$Ho_J = \{a \in H : a(1 + a_{w_i}) = 0 \text{ for all } w_i \in J\}.$$

Ho_J has basis $\{a_w : w \in Y_{\hat{L}},\ \hat{L} \subseteq \hat{J}\}$, dimension $|X_J|$ and $Ho_J = \sum^{\oplus}_{J \subseteq L} He_L^{\wedge} o_L$. Further, the map $\theta : Ho_J \to He_J^{\wedge} o_J$ given by projection defines an isomorphism $\theta : {}^{Ho}J/\sum_{J \subset L} Ho_L \cong He_J^{\wedge} o_J$.

Finally, $He_J^{\wedge} o_J = \{a \in Ho_J : ao_L = 0 \text{ for all } L \supset J\}$.

<u>Proof</u>: Clearly $Ho_J \leq \{a \in H : a(1 + a_{w_i}) = 0 \text{ for all } w_i \in J\}$. Conversely take $a \in H$ such that $a(1 + a_{w_i}) = 0$ for all $w_i \in J$.

Then $aa_{w_i} = -a$ for all $w_i \in J$; in particular,

$aa_{w_{oJ}} = (-1)^{l(w_{oJ})} a$, and so $a = ao_J$. Hence $a \in Ho_J$, and

$$Ho_J = \{a \in H : a(1 + a_{w_i}) = 0 \text{ for all } w_i \in J\}.$$

Let $a = \sum_{w \in W} u_w a_w \in Ho_J$, $u_w \in K$. Then $a(1 + a_{w_i}) = 0$ for all $w_i \in J$; thus

$$\sum_{w \in W} u_w a_w + \sum_{\substack{w \in W \\ w(r_i) > 0}} u_w a_{ww_i} - \sum_{\substack{w \in W \\ w(r_i) < 0}} u_w a_w = 0 \text{ for all } w_i \in J.$$

If $w(r_i) > 0$, the coefficient of a_w on the left side is u_w, so $u_w = 0$ if $w(r_i) > 0$. If $w(r_i) < 0$, the coefficient of a_w on the left side is $u_w + u_{ww_i} - u_w = u_{ww_i} = 0$ as $ww_i(r_i) > 0$. Hence $a = \sum_{\substack{w \in W \\ w(\prod_J) < 0}} u_w a_w = \sum_{\hat{L} \subseteq \hat{J}} \sum_{y \in Y_{\hat{L}}} u_y a_y$.

Conversely, if $y \in \{w \in W : w \in Y_{\hat{L}}, \hat{L} \subseteq \hat{J}\}$ then $a_y \in Ho_J$. Thus $\{a_y : y \in Y_{\hat{L}}, \hat{L} \subseteq \hat{J}\}$ is a basis of Ho_J. So $\dim Ho_J = \sum_{\hat{L} \subseteq \hat{J}} |Y_{\hat{L}}| = |X_J|$. Obviously $\sum^{\oplus}_{J \subseteq L} He_{\hat{L}} o_L \leqslant Ho_J$, and considering dimensions they must be equal. Hence $\{a_y e_{\hat{L}} o_L : y \in Y_{\hat{L}} : L \supseteq J\}$ is also a basis of Ho_J.

The rest of the proof follows similar lines as (4.4.15), and uses the proof of (4.4.9(2)).

(4.4.17) LEMMA: Let ψ_J be the character of the representation of H on $Ho_{\hat{J}} e_J$. Then ψ_J takes values as follows: For each $w \in W$, let $w = w_{i_1} \ldots w_{i_t}$ be a reduced expression for w, and set $J(w) = \{w_{i_j} : 1 \leqslant j \leqslant t\}$. Then

$$\psi_J(a_w) = (-1)^{l(w)} N_J(w),$$

where $N_J(w)$ = the number of elements $y \in Y_J$ such that $y^{-1}(\Pi_{J(w)}) \subseteq \Phi^-$

Proof: Use (4.4.10).

(4.4.18a) LEMMA: Let ϕ_J be the character of the representation of H on He_J. Then ϕ_J takes values as follows: for $w \in W$ let $w = w_{i_1}..w_{i_t}$ be a reduced expression for w. Set $J(w) = \{w_{i_j}: 1 \leq j \leq t\}$. Then

$$\phi_J(a_w) = (-1)^{l(w)} M_J(w)$$

where $M_J(w)$ = the number of elements $x \in X_J$ such that $x^{-1}(\Pi_{J(w)}) \subseteq \Phi^-$. Also, $M_J(w) = \sum_{J \subseteq L} N_L(w)$.

Proof: He_J has basis $\{a_w e_J : w \in X_J\}$. For any $w_i \in R$,

$$a_{w_i} a_w e_J = \begin{cases} -a_w e_J \text{ if } w^{-1}(r_i) < 0 \\ a_{w_i w} e_J \text{ where } w_i w \in X_J \text{ if } w^{-1}(r_i) > 0, \\ \qquad \text{and } w^{-1}(r_i) \neq r_j \text{ for any } r_j \in \Pi. \\ 0 \text{ if } w^{-1}(r_i) = r_j \text{ for some } r_j \in \Pi_J, \text{ for then} \\ \qquad a_{w_i} a_w = a_w a_{w_j}, \text{ and } a_{w_j} e_J = 0 \end{cases}$$

The result now follows.

(4.4.18b) LEMMA: Let μ_J be the character of the representation of H on Ho_J. Then μ_J takes values as follows: for each $w \in W$, let $w = w_{i_1}..w_{i_t}$ be a reduced expression for w, and set $J(w) = \{w_{i_j}: 1 \leq j \leq t\}$. Then

$$\mu_J(a_w) = (-1)^{l(w)} L_J(w)$$

where $L_J(w)$ = the number of elements $z \in Z_J$ such that $z^{-1}(\prod_{J(w)}) \subsetneq \phi^-$, and $Z_J = \{w \in W : w(\prod_J) \subsetneq \phi^-\}$. Note that $Z_J = \bigcup_{L \subseteq \hat{J}} Y_L$.

<u>Proof</u>: Ho_J has basis $\{a_w : w \in Z_J\}$. For all $w_i \in R$,

$$a_{w_i} a_w = \begin{cases} -a_w & \text{if } w^{-1}(r_i) < 0 \\ a_{w_i w} & \text{if } w^{-1}(r_i) > 0 \end{cases}$$

If $w \in Z_J$, $w_i \in R$ and $w^{-1}(r_i) > 0$, then $w_i w \in Z_J$, for if $r_j \in \prod_J$, $w(r_j) = -s$ for some $s \in \phi^+$, and $w_i(s) < 0$ if and only if $s = r_i$. But if $s = r_i$, $w^{-1}(r_i) = -r_j$ - impossible. The result now follows.

(4.4.19) <u>COROLLARY</u>: (1) $\phi_J = \sum_{J \subseteq L} \Psi_L$ for all $J \subseteq R$.

(2) $\mu_J = \sum_{J \subseteq L} \Psi_{\hat{L}}$ for all $J \subseteq R$.

A direct sum decomposition of H into indecomposable left ideals is equivalent to expressing the identity of H as a sum of mutually orthogonal primitive idempotents. Let $1 = \sum_{J \subseteq R} q_J$ and $1 = \sum_{J \subseteq R} p_J$ be the decompositions of 1 corresponding to the decompositions $H = \sum^{\oplus}_{J \subseteq R} Ho_{\hat{J}} e_J$ and $H = \sum^{\oplus}_{J \subseteq R} He_J o_{\hat{J}}$ respectively, where $Hq_J = Ho_{\hat{J}} e_J$ and $Hp_J = He_J o_{\hat{J}}$. There does not appear to be a specific expression for the q_J or the p_J in terms of $\{a_y o_{\hat{J}} e_J : y \in Y_J\}$ or $\{a_y e_J o_{\hat{J}} : y \in Y_J\}$ respectively, but in Appendix 3 we give some examples of $\{q_J\}$ and $\{p_J\}$.

(4.4.20) THEOREM: Let $\{q_J : J \subseteq R\}$ be a set of mutually orthogonal primitive idempotents with $q_J \in Ho_{\hat{J}}e_J$ for all $J \subseteq R$ such that $1 = \sum_{J \subseteq R} q_J$. Then $Ho_{\hat{J}}e_J = Hq_J$, and if N is the nilpotent radical of H, $No_{\hat{J}}e_J = Nq_J$ is the unique maximal left ideal of Hq_J, and ${}^{Hq_J}/{}_{Nq_J} \cong K$. ${}^{Hq_J}/{}_{Nq_J}$ affords the representation λ_J of H defined in (4.3.1). Finally, ${}^{H}/{}_{N} = \sum^{\oplus}_{J \subseteq R} {}^{Hq_J}/{}_{Nq_J} \cong K \oplus K \oplus .. \oplus K$, 2^n summands, where $n = |R|$.

Proof: By the theory of Artinian rings, Nq_J is the unique maximal left ideal of Hq_J, and ${}^{H}/{}_{N} = \sum^{\oplus}_{J \subseteq R} {}^{Hq_J}/{}_{Nq_J}$. Since $q_J \in Ho_{\hat{J}}e_J$, $Hq_J \leqslant Ho_{\hat{J}}e_J$. As $H = \sum^{\oplus}_{J \subseteq R} Hq_J = \sum^{\oplus}_{J \subseteq R} Ho_{\hat{J}}e_J$, we must have $Hq_J = Ho_{\hat{J}}e_J$ for all $J \subseteq R$. Then $Nq_J = NHq_J = NHo_{\hat{J}}e_J = No_{\hat{J}}e_J$ is the unique maximal left ideal of Hq_J. But $\{\sum_{\substack{y \in Y_J \\ y \neq w_{o\hat{J}}}} u_y a_y o_{\hat{J}} e_J : u_y \in K\}$ is a maximal left ideal of $Ho_{\hat{J}}e_J$, by looking at (4.4.10), and so $Nq_J = \{\sum_{\substack{y \in Y_J \\ y \neq w_{o\hat{J}}}} u_y a_y o_{\hat{J}} e_J : u_y \in K\}$. Then ${}^{Hq_J}/{}_{Nq_J}$ is a one-dimensional H-module generated by $a_{w_{o\hat{J}}} o_{\hat{J}} e_J + Nq_J$ which affords the representation λ_J of H, and since every element of ${}^{Hq_J}/{}_{Nq_J}$ is of the form $ka_{w_{o\hat{J}}} o_{\hat{J}} e_J + Nq_J$ for some $k \in K$, ${}^{Hq_J}/{}_{Nq_J} \cong K$ for all $J \subseteq R$. Hence the result.

(4.4.21) THEOREM: Let $\{p_J : J \subseteq R\}$ be a set of mutually orthogonal primitive idempotents with $p_J \in He_J o_{\hat{J}}$ for all $J \subseteq R$ such that $1 = \sum_{J \subseteq R} p_J$. Then $He_J o_{\hat{J}} = Hp_J$, and if N is the nilpotent radical of H, $Ne_J o_{\hat{J}} = Np_J$ is the unique maximal left ideal of Hp_J, and $Hp_J/Np_J \cong K$. Hp_J/Np_J affords the representation λ_J of H defined in (4.3.1). Finally, $H/N = \sum_{J \subseteq R}^{\oplus} Hp_J/Np_J \cong K \oplus K \oplus \ldots \oplus K$, 2^n summands, where $n = |R|$.

Proof: We have similar relations to (4.4.10) for the elements $a_y e_J o_{\hat{J}}$ with $y \in Y_J$; they are as follows:

(4.4.10)' Let $y \in Y_J$ and let $w_i \in R$. Then

$$a_{w_i} a_y e_J o_{\hat{J}} = \begin{cases} -a_y e_J o_{\hat{J}} \text{ if } y^{-1}(r_i) < 0 \\ 0 \text{ if } y^{-1}(r_i) = r_j \text{ for some } r_j \in \prod_J \\ a_{w_i y} e_J o_{\hat{J}} \text{ where } w_i y \in Y_J \text{ if } y^{-1}(r_i) > 0 \\ \qquad \text{and } y^{-1}(r_i) \neq r_j \text{ for any } r_j \in \prod_J . \end{cases}$$

The result now follows using the proof of (4.4.20) since $\{ \sum_{\substack{y \in Y_J \\ y \neq w_{o\hat{J}}}} \upsilon_y a_y e_J o_{\hat{J}} : \upsilon_y \in K\}$ is a maximal left ideal of $He_J o_{\hat{J}}$.

(4.4.22) LEMMA: $\{ka_{w_o w_{oJ}} o_{\hat{J}} e_J : k \in K\}$ and $\{ka_{w_o w_{oJ}} e_J o_{\hat{J}} : k \in K\}$ are minimal submodules of $Ho_{\hat{J}} e_J$ and $He_J o_{\hat{J}}$ respectively, where $w_o w_{oJ}$ is the unique element of maximal length in Y_J. These minimal left ideals both afford the representation

$\lambda_{\bar{J}}$ of H, where $\bar{J} = \{w_i \in R$: there exists $w_j \in J$ with $w_o w_j = w_i w_o\}$, or, alternatively, $\Pi_{\bar{J}}$ is defined by $w_o(\Pi_J) = -\Pi_{\bar{J}}$.

<u>Proof</u>: The submodules given are clearly minimal by (1.3.5) and (1.3.7). Since $w_o w_{oJ} = w_{o\bar{J}} w_o$, these minimal left ideals both afford the representation $\lambda_{\bar{J}}$ of H as $(w_{o\bar{J}} w_o)^{-1} = w_o w_{o\bar{J}} \in Y_{\bar{J}}$.

Examples:

(1) H of type $(W(A_1),\{w_1\})$

$$H = \{k_1 1 + k_2 a_{w_1} : k_1, k_2 \in K\}$$

J	e_J	o_J
$\emptyset$	1	1
$\{w_1\}$	$1 + a_{w_1}$	$-a_{w_1}$

Then $H = H(1 + a_{w_1}) \oplus H(-a_{w_1})$, where $H(1 + a_{w_1})$ has basis $\{(1 + a_{w_1})\}$, and $H(-a_{w_1})$ has basis $\{a_{w_1}\}$. Notice that in this case the two decompositions are identical, i.e. $o_\emptyset e_{\{w_1\}} = e_{\{w_1\}} o_\emptyset$ and $o_{\{w_1\}} e_\emptyset = e_\emptyset o_{\{w_1\}}$.

(2) H of type $(W(A_2), \{w_1,w_2\})$.

H has K-basis $1, a_{w_1}, a_{w_2}, a_{w_1w_2}, a_{w_2w_1}, a_{w_1w_2w_1}$.

J	e_J	o_J
$\emptyset$	1	1
$\{w_1\}$	$1 + a_{w_1}$	$-a_{w_1}$
$\{w_2\}$	$1 + a_{w_2}$	$-a_{w_2}$
$\{w_1,w_2\}$	$(1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1})$	$-a_{w_1w_2w_1}$.

Then $H = H(-a_{w_1w_2w_1}) \oplus H(-a_{w_1})(1 + a_{w_2}) \oplus H(-a_{w_1})(1 + a_{w_2})$

$$\oplus\, H(1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1})$$

and $H = H(-a_{w_1w_2w_1}) \oplus H(1 + a_{w_2})(-a_{w_1}) \oplus H(1 + a_{w_1})(-a_{w_2})$

$$\oplus\, H(1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1})$$

where $H(-a_{w_1w_2w_1})$ has K-basis $\{a_{w_1w_2w_1}\}$, $H(-a_{w_1})(1 + a_{w_2})$

has K-basis $\{a_{w_1}(1 + a_{w_2}),\ a_{w_2w_1}(1 + a_{w_2})\}$,

$H(-a_{w_2})(1 + a_{w_1})$ has K-basis $\{a_{w_2}(1 + a_{w_1}),\ a_{w_1w_2}(1 + a_{w_1})\}$,

$H(1 + a_{w_2})(-a_{w_1})$ has K-basis $\{a_{w_1} - a_{w_1w_2w_1},\ a_{w_2w_1} + a_{w_1w_2w_1}\}$,

$H(1 + a_{w_1})(-a_{w_2})$ has K-basis $\{a_{w_2} - a_{w_1w_2w_1},\ a_{w_1w_2} + a_{w_1w_2w_1}\}$,

and $H(1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1})$ has K-basis

$\{(1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1})\}$. Also,

$H(-a_{w_1})(1 + a_{w_2}) \cong H(1 + a_{w_2})(-a_{w_1})$ as left ideals and

$H(-a_{w_2})(1 + a_{w_1}) \cong H(1 + a_{w_1})(-a_{w_2})$ as left ideals.

(4.4.23) <u>NOTE</u>:

(1) By the same methods we also have that

$$H = \sum_{J \subseteq R}^{\oplus} e_J o_{\hat{J}} H \qquad \text{and} \qquad H = \sum_{J \subseteq R}^{\oplus} o_{\hat{J}} e_J H,$$

both being direct sum decompositions of H into 2^n right ideals, where $n=|R|$. Further, we have that $e_J o_{\hat{J}} H$ has K-basis $\{e_J o_{\hat{J}} a_{w^{-1}}: w \in Y_J\}$, and that $o_{\hat{J}} e_J H$ has K-basis $\{o_{\hat{J}} e_J a_{w^{-1}}: w \in Y_J\}$. All the results for the left ideals He_J, Ho_J, $He_J o_{\hat{J}}$ and $Ho_{\hat{J}} e_J$ have analogues for the right ideals $e_J H$, $o_J H$, $o_{\hat{J}} e_J H$ and $e_J o_{\hat{J}} H$ respectively.

(2)Suppose $y \in Y_{\hat{J}}$; then $y =(yw_{oJ})w_{oJ}$ with $l(y)=l(yw_{oJ})+l(w_{oJ})$. Then $a_y e_J o_{\hat{J}} = a_{yw_{oJ}} a_{w_{oJ}} e_J o_{\hat{J}} = 0$ unless $J=\emptyset$. Suppose there exists an element $w_i \in J$ such that $w_i w_j = w_j w_i$ for all $w_j \in W_{\hat{J}}$. Then $a_y o_{\hat{J}} e_J = 0$. So we cannot get similar results using the elements $a_y e_J o_{\hat{J}}$ and $a_y o_{\hat{J}} e_J$ where $y \in Y_{\hat{J}}$.

(4.5) <u>The Cartan Matrix of H.</u>

We have that $H = \sum^{\oplus}_{J \subseteq R} U_J$, where $U_J = H o^{\wedge}_J e_J$ is an indecomposable left H-module. Thus $\{U_J : J \subseteq R\}$ are the principal indecomposable H-modules. $\{{}^{U}J/\text{rad } U_J : J \subseteq R\}$, where rad U_J is the unique maximal submodule of U_J, are irreducible H-modules, such that $M_J = {}^{U}J/\text{rad } U_J$ affords the representation λ_J of H.

<u>Definition:</u> The Cartan matrix C of H, where H is of type (W,R), with $|R| = n$, is a $2^n \times 2^n$ matrix with rows and columns indexed by the subsets of R, and if we write $C = (c_{JL})$, then

c_{JL} = the number of times M_L is a composition factor of U_J.

(4.5.1) <u>THEOREM</u>: For all $J,L \subseteq R$,

$$c_{JL} = |Y_J \cap (Y_L)^{-1}| = |Y_L \cap (Y_J)^{-1}| = c_{LJ}.$$

Hence C is a symmetric matrix.

<u>Proof</u>: U_J has K-basis $\{a_y o^{\wedge}_J e_J = a_y e_J : y \in Y_J\}$. Let $y_1, \ldots y_s$ be all the elements of Y_J written in order of increasing length; if $i > j$ then $l(y_i) \geqslant l(y_j)$. Then set $U_J(i) = \{\sum_{j \geqslant i} k_{y_j} a_{y_j} e_J : k_{y_j} \in K\}$. $U_J(i)$ is a left ideal of H for all i, and $U_J(i) > U_J(i+1)$ for all i, $1 \leqslant i \leqslant s-1$. Then $U_J = U_J(1) > U_J(2) > \ldots > U_J(s) > 0$ is a composition series of U_J, with ${}^{U_J(i)}/U_J(i+1)$ being an irreducible

H-module with basis $a_{y_i}e_J + U_J(i+1)$ and affording the irreducible representation λ_L, defined in (4.3.1), where we determine L as follows: recall (4.4.10).

Let $w_j \in R$ and $y_i \in Y_J$; then

$$a_{w_j}a_{y_i}e_J = \begin{cases} -a_{y_i}e_J \text{ if } y_i^{-1}(r_j) < 0 \\ 0 \text{ if } y_i^{-1}(r_j) = r_k \text{ for some } r_k \in \Pi \\ a_{w_jy_i}e_J \text{ where } w_jy_i = y_l \text{ for some } y_l \in Y_J \\ \qquad \text{with } i < l, \text{ if } y_i^{-1}(r_j) > 0 \text{ but} \\ \qquad y_i^{-1}(r_j) \neq r_k \text{ for any } r_k \in \Pi \end{cases}$$

Hence λ_L: $a_{w_j} \to \begin{cases} -1 \text{ if } y_i^{-1}(r_j) < 0 \\ 0 \text{ if } y_i^{-1}(r_j) > 0 \end{cases}$

That is, $y_i^{-1} \in Y_L$.

Hence c_{JL} = the number of elements $y \in Y_J$ such that $y^{-1} \in Y_L$.

$$= |Y_J \cap (Y_L)^{-1}| = |Y_L \cap (Y_J)^{-1}|$$

since if $y \in Y_J \cap (Y_L)^{-1}$, then $y^{-1} \in Y_L \cap (Y_J)^{-1}$.

Some examples of the Cartan matrix for various types of H are given in Appendix 4.

<u>Definition</u>: Let E be a centrally primitive idempotent in H. Then the block B = B(E) is the class of all finitely generated H-modules V satisfying EV = V.

Since $1 = \sum_{i=1}^{t} e_i$, a sum of pairwise orthogonal centrally primitive idempotents in H, then any H-module V can be written $V = 1.V = \sum_{i=1}^{t\oplus} e_iV$. So if V is indecomposable,

$V = e_i V$ for some e_i and $V \in B(e_i)$. Thus every non-zero finitely generated indecomposable H-module is in a unique block. Furthermore, if $V \in B = B(e)$, then for all $v \in V$, $ev = v$.

<u>Definition</u>: Let e_1 and e_2 be primitive idempotents in H. Then we say e_1 and e_2 are linked if there is a sequence $e_1 = e_{i_1}, e_{i_2}, \ldots e_{i_n} = e_2$ of primitive idempotents such that for each j, He_{i_j} and $He_{i_{j+1}}$ have a common irreducible component.

(4.5.2) <u>LEMMA</u>: The primitive idempotents e_1 and e_2 of H are linked if and only if He_1 and He_2 are in the same block.

<u>Proof</u>: See Dornhoff [12], theorem 46.2.

<u>Definition</u>: If e is a centrally primitive idempotent in H and $B = B(e)$ is the corresponding block, then the Cartan matrix of the algebra $He = eHe$ is the Cartan matrix of the block B.

(4.5.3) <u>THEOREM</u> (Dornhoff [12], theorem 46.3): Let A be a finite-dimensional algebra over the field K, and let $B_1, \ldots B_m$ be all of the distinct blocks of A. Let C be the Cartan matrix of A, C_i the Cartan matrix of B_i. Then

(1) with a suitable arrangement of rows and columns,

$$C = \begin{bmatrix} C_1 & & & (0) \\ & C_2 & & \\ & & \ddots & \\ (0) & & & C_m \end{bmatrix}.$$

(2) for any $i, 1 \leq i \leq m$, it is impossible to arrange the

rows and columns of C_i so that $C_i = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ for some matrices A and B.

(4.5.4) <u>THEOREM</u>: Let H be the 0-Hecke algebra over the field K of type (W,R), where W is indecomposable. Then if $|R| > 1$, H has three blocks. If $|R| = 1$, then H has two blocks.

<u>Proof</u>: If $|R| = 1$, then $W = W(A_1)$ and $H = H(1 + a_{w_1}) \oplus H(-a_{w_1})$, where $R = \{w_1\}$. Both $(1 + a_{w_1})$ and $(-a_{w_1})$ are primitive idempotents as well as being central. Hence H has only two blocks.

Now suppose that $|R| > 1$. $\left\lfloor 1 + a_{w_0} \right\rceil = e_R$ and $(-1)^{l(w_0)} a_{w_0}$ are primitive and centrally primitive idempotents in H and so correspond to two distinct blocks. The other primitive idempotents in H, i.e. $\{q_J : J \neq \emptyset, R\}$ as in (4.4.20), determine at least one other block. We will show that provided W is indecomposable the Cartan matrix C' corresponding to the indecomposables U_J for $J \neq \emptyset, R$ and the irreducibles M_L for $L \neq \emptyset, R$ cannot be expressed in the form $C' = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$.

Suppose that C' can be put in the form above. Let $S_1 = \{J \subset R : U_J$ and M_J index the rows and columns of $C_1\}$, $S_2 = \{J \subset R : U_J$ and M_J index the rows and columns of $C_2\}$. Suppose for some $J \subset R$, $|J| = n-1$ (where $n = |R|$), that $J \in S_1$. Then we show

(1) for all $L \subset R$ with $|L|=n-1$, $L \in S_1$.

(2) by decreasing induction on $|J|$ for all $J \neq \emptyset, R$ that $J \in S_1$.

(a) Suppose $J = \{w_1,..\hat{w}_j,..w_n\}$ and $L = \{w_1,..\hat{w}_{j+1},..w_n\}$, where the nodes corresponding to w_j and w_{j+1} in the graph of W are joined. Then the order of $w_j w_{j+1}$ is greater than 2. Now $w_{o\hat{J}} = w_j \in Y_J$ and $w_{o\hat{L}} = w_{j+1} \in Y_L$. Since the order of $w_j w_{j+1}$ is greater than 2, we have that $w_{j+1}w_j \in Y_J$ and $w_j w_{j+1} \in Y_L$; that is, $w_{j+1}w_j \in Y_J \cap (Y_L)^{-1}$. Hence $J \in S_1$ if and only if $L \in S_1$.

Hence if there is some $J \in S_1$, with $|J| = n-1$, then all L with $|L| = n-1$ are in S_1 by the above.

(b) Suppose that for all $J \subset R$ with $|J| > m$ that $J \in S_1$. Choose $L \subset R$ with $|L| = m$. We show $L \in S_1$. Suppose $L = \{w_{i_1},..w_{i_m}\}$ with $1 \leq i_1 < i_2 < .. < i_m \leq n$. Since W is indecomposable, $L \neq \emptyset, R$ then $|Y_L| > 1$. Choose some $w_{i_j} \in L$ and $w_k \in \hat{L}$ such that $w_{i_j}w_k$ has order r, where $r \geqslant 3$. Then $w_{i_j}w_{o\hat{L}} \in Y_L$ (as $w_{o\hat{L}}(r_{i_j}) \neq r_i$ for any $r_i \in \prod_L$, since otherwise $w_{o\hat{L}}(r_{i_j}) = r_i$ for some $r_i \in \prod_L$ implies that $r_{i_j} = r_i$ and $w_{o\hat{L}}$ is a product of reflections corresponding to roots orthogonal to r_{i_j}, and so for all $w_k \in \hat{L}$, $w_{i_j}w_k = w_k w_{i_j}$ - contradiction). Now consider $(w_{i_j}w_{o\hat{L}})^{-1} = w_{o\hat{L}}w_{i_j}$. Suppose $w_{i_l} \in L$, $w_{i_l} \neq w_{i_j}$. Then $w_{o\hat{L}}w_{i_j}(r_{i_l}) \in \phi^+$. Also $w_{o\hat{L}}w_{i_j}(r_{i_j}) \in \phi^-$. Suppose $w_k \in \hat{L}$.

Then $w_{o\hat{L}}w_{i_j}(r_k) = w_{o\hat{L}}(r_k + ur_{i_j})$ with $u \geq 0$

$$= w_{o\hat{L}}(r_k) + uw_{o\hat{L}}(r_{i_j}).$$

If $u=0$, i.e. if $w_{i_j}w_k = w_kw_{i_j}$, then $w_{o\hat{L}}w_{i_j}(r_k) \in \phi^-$.

If $u > 0$, as $w_{o\hat{L}}(r_k) = -r_i$ for some $r_i \in \prod_{\hat{L}}$, and $w_{o\hat{L}}(r_{i_j}) \in \phi^+$, $w_{o\hat{L}}(r_{i_j}) \neq r_{i_s}$ for any $r_{i_s} \in \prod_L$, we have $w_{o\hat{L}}w_{i_j}(r_k) \in \phi^+$. Hence $w_{o\hat{L}}w_{i_j} \in Y_M$, where

$$M = \{L - \{w_{i_j}\}\} \cup \{w_k \in \hat{L}\colon w_{i_j}w_k \text{ has order} > 2\}$$

$= \{L - \{w_{i_j}\}\} \cup \{w_k \in \hat{L}$: the node corresponding to w_k in the graph of W is joined to that corresponding to $w_{i_j}\}$.

Now $|M| > |L|$ if the node corresponding to w_{i_j} is joined to at least two nodes corresponding to elements of $\hat{L}$, and then $L \in S_1$ by induction.

Let P_i be the node of the graph of W which corresponds to $w_i \in R$, $1 \leq i \leq n$. Then suppose P_{i_j} is joined to only one P_k for all $w_k \in \hat{L}$. Then the above argument shows that $L = \{w_{i_1}, \ldots w_{i_m}\}$ and $M = \{w_{i_1}, \ldots \hat{w}_{i_j}, \ldots w_{i_m}, w_k\}$ belong to the same S_i, where $i=1$ or $i=2$. Since $|L| \leq n-2$, $|\hat{L}| \geq 2$. Let w_{k_1} and w_{k_2} be any two elements of $\hat{L}$, such that there exists a sequence $P_{k_1} = P_{j_o}, P_{j_1}, \ldots P_{j_r} = P_{k_2}$ of nodes such that P_{j_i} and $P_{j_{i+1}}$ are joined for all i, $0 \leq i \leq r-1$, and P_{j_i} corresponds to an element of L for all i, $1 \leq i \leq r-1$. If $r = 1$, then P_{k_1} and P_{k_2} are joined. Without loss of generality, we may suppose there exists $w_{i_s} \in L$ such that P_{i_s} is joined to P_{k_1}. Then let $M = \{L - \{w_{i_s}\}\} \cup \{w_{k_1}\}$. M and L belong to the same S_i, and by the above, as M has an element w_{k_1} such that $w_{k_1}w_{i_s}$ and $w_{k_1}w_{k_2}$ both have order > 2, where $w_{i_s}, w_{k_2} \in \hat{M}$, $w_{i_s} \neq w_{k_2}$, then $M \in S_1$.

If $r = 2$, then L and M are in the same S_i, where $M = \{L - \{w_{j_1}\}\} \cup \{w_{k_1}, w_{k_2}\}$, and by induction $M \in S_1$.

If $r > 2$, define $L_o = L$

$$L_1 = \{L - \{w_{j_1}\}\} \cup \{w_{j_o}\},$$

$$\cdots\cdots$$

$$L_{r-2} = \{L_{r-3} - \{w_{j_{r-2}}\}\} \cup \{w_{j_{r-3}}\}$$

Then L_o, $L_1, \ldots L_{r-2}$ are all in the same S_i, and by the above, $L_{r-2} \in S_1$.

Hence $L \in S_1$. Then $S_2 = \emptyset$, and so H has precisely three blocks.

(4.5.5) <u>THEOREM</u>: Let H be a 0-Hecke algebra of type (W,R). Suppose W is decomposable, and let $W = W_1 \times W_2 \times \ldots \times W_r$, where each W_i is an indecomposable Coxeter group, and the corresponding Coxeter system is (W_i,R_i). Let H_i be the 0-Hecke algebra of type (W_i,R_i), and let m_i be the number of blocks of H_i. Then H has $m_1m_2\ldots m_r$ blocks.

<u>Proof</u>: Suppose that $1 = \sum_{i=1}^{t} e_i$ where the e_i are mutually orthogonal centrally primitive idempotents in H. Then the number of blocks of H is equal to t.

Now for all $w \in W_i$, $w' \in W_j$, where $1 \leq i,j \leq r$ and $i \neq j$ we have that $a_w a_{w'} = a_{ww'} = a_{w'w} = a_{w'} a_w$, and so it follows that if f_i is a central primitive idempotent of H_i, then $f_1 \ldots f_r$ is a central primitive idempotent of H. Suppose $1_{H_i} = \sum_{j=1}^{t(i)} f_{ij}$ where for a fixed i, $\{f_{ij} : 1 \leq j \leq t(i)\}$ is a set of mutually orthogonal central primitive idempotents in H_i. Then $1_H = \sum_{j_1=1}^{t(1)} \ldots \sum_{j_r=1}^{t(r)} f_{1j_1} \ldots f_{rj_r}$, a sum of mutually orthogonal central primitive idempotents in H, and so H has $t(1)t(2) \ldots t(r)$ blocks, where $t(i) = m_i$.

Chapter 5: DECOMPOSITIONS OF THE GENERIC RING.

Let $A = A_{B_o}(u)$ be the generic ring of the system S of finite groups with (B,N) pairs of type (W,R), where $B_o = \{g(u)/h(u): g(u), h(u) \in Q[u], u \nmid h(u)\}$. A is the associative algebra over B_o with basis $\{a_w : w \in W\}$, and multiplication is given by the following: for all $w_i \in R$ and all $w \in W$, we have

$$a_{w_i} a_w = \begin{cases} a_{w_i w} & \text{if } l(w_i w) > l(w) \\ u^{c_i} a_{w_i w} + (u^{c_i} - 1) a_w & \text{if } l(w_i w) < l(w) \end{cases}$$

Also, if $w = w_{i_1} \cdots w_{i_s}$ is a reduced expression for $w \in W$, then $a_w = a_{w_{i_1}} \cdots a_{w_{i_s}}$.

Now each group $G(q) \in S$ has a parabolic subgroup $G_J(q)$ for each subset $J \subseteq R$, and $G_J(q)$ is itself a finite group with a (B,N) pair of type (W_J,J).

(5.1) <u>Definition</u>: Let $A_J = A_{J,B_o}(u)$ be the generic ring of the system S_J of finite groups with (B,N) pairs of type (W_J,J) with $\mathcal{P}$ the same as for S, and $\{c_i : w_i \in J\}$ are the same as in S. Each $G(q) \in S$ determines a $G_J(q) \in S_J$.

(5.2) <u>Definition</u>: For all $w \in W$, $w \neq 1$, let $w = w_{i_1} \cdots w_{i_s}$ be a reduced expression for w, and define

$$c_w = c_{i_1} + \cdots + c_{i_s}.$$

If $w=1$, let $c_1 = 0$. (see also (3.2.3)).

(5.3) <u>LEMMA</u>: c_w is independent of the reduced expression for w, and since all c_i are positive integers, c_w is a positive integer for all $w \in W$, $w \neq 1$.

Proof: Since we can get from one reduced expression for w to another by substitutions of the form

$$(w_i w_j w_i \ldots)_{n_{ij}} = (w_j w_i w_j \ldots)_{n_{ij}}$$

where n_{ij} = the order of $w_i w_j$ in W, we need to show

$c_i + c_j + c_i + \ldots (n_{ij}$ terms$) = c_j + c_i + c_j + \ldots (n_{ij}$ terms$)$.

If n_{ij} is even this is obvious. If n_{ij} is odd, then w_i and w_j are conjugate in $W_{\{w_i, w_j\}}$ and so $c_i = c_j$, and again the result is obvious.

(5.4) Definition: The characteristic function of S_J for all $J \subseteq R$ is the polynomial

$$f(u)_J = \sum_{w \in W_J} u^{c_w} \in Z[u].$$

(Compare (3.2.4)). $f(u)_\emptyset = 1$.

(5.5) Definition: For each $J \subseteq R$, define

(1) $e_J = \dfrac{1}{f(u)_J} \displaystyle\sum_{w \in W_J} a_w$.

(2) $o_J = \dfrac{1}{f(u)_J} \displaystyle\sum_{w \in W_J} (-1)^{l(w)} u^{c_{ww_{oJ}}} a_w$ where

w_{oJ} is the unique element of maximal length in W_J.

e_J and o_J are elements of A_J. We will now show that they are central idempotents in A_J.

(5.6) LEMMA: (1) For all $w_i \in J$, $a_{w_i} e_J = u^{c_i} e_J = e_J a_{w_i}$.

(2) For all $w_i \in J$, $a_{w_i} o_J = -o_J = o_J a_{w_i}$.

Proof: (1) $a_{w_i} e_J = \frac{1}{f(u)_J} \sum_{w \in W_J} a_{w_i} a_w$

$$= \frac{1}{f(u)_J} \sum_{\substack{w \in W_J \\ w^{-1}(r_i) > 0}} a_{w_i w}$$

$$+ \frac{1}{f(u)_J} \sum_{\substack{w \in W_J \\ w^{-1}(r_i) < 0}} \{u^{c_i} a_{w_i w} + (u^{c_i} - 1) a_w\}$$

$$= \frac{1}{f(u)_J} \sum_{\substack{w \in W_J \\ w^{-1}(r_i) > 0}} \{a_{w_i w} + u^{c_i} a_w + (u^{c_i} - 1) a_{w_i w}\}$$

$$= u^{c_i} e_J.$$

Similarly $e_J a_{w_i} = u^{c_i} e_J$.

(2) $$a_{w_i} o_J = \frac{1}{f(u)_J} \sum_{\substack{w \in W_J \\ w^{-1}(r_i) > 0}} (-1)^{l(w)} u^{c_{ww_{oJ}}} a_{w_i w}$$

$$+ \frac{1}{f(u)_J} \sum_{\substack{w \in W_J \\ w^{-1}(r_i) < 0}} (-1)^{l(w)} u^{c_{ww_{oJ}}} \{u^{c_i} a_{w_i w} + (u^{c_i} - 1) a_w\}$$

Suppose $ww_{oJ} = w_{i_1} \cdots w_{i_s}$, $l(ww_{oJ}) = s$, and $w^{-1}(r_i) < 0$.

Then $(ww_{oJ})^{-1}(r_i) > 0$ (as $r_i \in \Pi_J$), and so

$w_i ww_{oJ} = w_i w_{i_1} \cdots w_{i_s}$, $l(w_i ww_{oJ}) = s+1$. Hence

$c_{w_i ww_{oJ}} = c_i + c_{ww_{oJ}}$. So

$$a_{w_i} o_J = \frac{1}{f(u)_J} \sum_{\substack{w \in W_J \\ w^{-1}(r_i) > 0}} (-1)^{l(w)} \{u^{c_{ww_{oJ}}} a_{w_i w} - u^{c_{ww_{oJ}}} a_w - (u^{c_{ww_{oJ}}} - u^{c_{w_i ww_{oJ}}}) a_{w_i w}\}.$$

Hence $a_{w_i} o_J = -o_J$. Similarly $o_J a_{w_i} = -o_J$.

(5.7) COROLLARY: e_J and o_J are idempotents in the centre of A_J, for all $J \subseteq R$.

Proof: Follows from (5.6).

(5.8) LEMMA: Let $J \subseteq L \subseteq R$, and let $X_J^L = \{y_1, .. y_s\}$, where $X_J^L = \{w \in W_L : w(\prod_J) \in \Phi^+\}$. (By (1.3.2) the set X_J^L is a set of left coset representatives for W_L mod W_J, and each element of W_L can be expressed uniquely in the form $y_i w_J$ for some i and some $w_J \in W$ with $l(y_i w_J) = l(y_i)+l(w_J)$).

Then (1) $e_L = \dfrac{f(u)_J}{f(u)_L} \sum_{i=1}^{s} a_{y_i} e_J = \dfrac{f(u)_J}{f(u)_L} \sum_{i=1}^{s} e_J a_{y_i^{-1}}$, and

$$(2)\quad o_L = \frac{f(u)_J}{f(u)_L} \sum_{i=1}^{s} (-1)^{l(y_i)} u^{c_{w_{oJ}w_{oL}} - c_{y_i}} a_{y_i} o_J$$

$$= \frac{f(u)_J}{f(u)_L} \sum_{i=1}^{s} (-1)^{l(y_i)} u^{c_{w_{oJ}w_{oL}} - c_{y_i}} o_J a_{y_i^{-1}} .$$

Proof: (1) $e_L = \dfrac{1}{f(u)_L} \sum_{w \in W_L} a_w = \dfrac{1}{f(u)_L} \left(\sum_{i=1}^{s} a_{y_i} \right) \left(\sum_{w \in W_J} a_w \right)$

$$= \frac{f(u)_J}{f(u)_L} \sum_{i=1}^{s} a_{y_i} e_J$$

and similarly for the other equality, as $\{y_1^{-1}, .. y_s^{-1}\}$ are a set of right coset representatives for W_L mod W_J.

$$(2)\quad o_L = \frac{1}{f(u)_L} \sum_{i=1}^{s} \sum_{w \in W_J} (-1)^{l(y_i)} (-1)^{l(w)} u^{c_{y_i w w_{oL}}} a_{y_i} a_w$$

Now let $y_i w w_{oL} = w_{i_1} .. w_{i_t}$, $l(y_i w w_{oL}) = t$. Since $w \in W_J$, $l(y_i w) = l(y_i) + l(w)$ and so $w w_{oL} = (y_i)^{-1} w_{i_1} .. w_{i_t}$, with $l(w w_{oL}) = l((y_i)^{-1}) + t$. Then

$$c_{ww_{oL}} = c_{y_i} + c_{y_i ww_{oL}} \text{ as } c_{(y_i)^{-1}} = c_{y_i}.$$

Now for all $w \in W_J$, $ww_{oL} = (ww_{oJ})(w_{oJ}w_{oL})$ and

$c_{ww_{oL}} = c_{ww_{oJ}} + c_{w_{oJ}w_{oL}}$. Hence

$c_{y_i ww_{oL}} = c_{ww_{oL}} - c_{y_i} = c_{ww_{oJ}} + c_{w_{oJ}w_{oL}} - c_{y_i}$. So

$$o_L = \frac{1}{f(u)_L} \sum_{i=1}^{s} \sum_{w \in W_J} (-1)^{l(y_i)} u^{(c_{w_{oJ}w_{oL}} - c_{y_i})} a_{y_i}$$

$$\times (-1)^{l(w)} u^{c_{ww_{oJ}}} a_w,$$

$$= \frac{f(u)_J}{f(u)_L} \sum_{i=1}^{s} (-1)^{l(y_i)} u^{(c_{w_{oJ}w_{oL}} - c_{y_i})} a_{y_i} o_J.$$

For the other equality, note that for $w \in W_J$,

$$c_{wy_i^{-1}w_{oL}} = c_{ww_{oJ}} + c_{w_{oJ}w_{oL}} - c_{y_i}.$$

(5.9) <u>COROLLARY</u>: For all $J \subseteq L \subseteq R$,

$$e_J e_L = e_L = e_L e_J$$

and $\quad o_J o_L = o_L = o_L o_J.$

<u>Proof</u>: Use (5.8) and (5.7).

(5.10) <u>LEMMA</u>: If J, $L \subseteq R$ and $J \cap L \neq \emptyset$, then $o_J e_L = 0$ and $e_J o_L = 0$.

<u>Proof</u>: Let $M = \{w_i\}$. Then

$$o_M e_M = \frac{1}{(f(u)_M)^2} (u^{c_{i_1}} - a_{w_i})(1 + a_{w_i})$$

$$= \frac{1}{(f(u)_M)^2} (u^{c_{i_1}} + u^{c_i} a_{w_i} - a_{w_i} - a_{w_i}^{\,2}) = 0$$

Similarly $e_M o_M = 0$. The result now follows using (5.8).

NOTE: From the definition of A, it is clear that for all $w, w' \in W$, we have $a_w a_{w'} = \sum_{w'' \in W} k_{w''} a_{w''}$, $k_{w''} \in \mathcal{O}$, summed over certain $w'' \in W$ for which

$$|l(w)-l(w')| \leqslant l(w'') \leqslant l(w)+l(w').$$

(5.11) LEMMA: (1) For all $y \in Y_J$,

$$a_y o_{\hat{J}} e_J = \frac{1}{f(u)_{\hat{J}} f(u)_J} a_{yw_{oJ}} + (\text{a linear combination of terms } a_w \text{ with } l(w) < l(yw_{oJ})\ ).$$

(2) For all $y \in Y_J$,

$$a_y e_J o_{\hat{J}} = \frac{1}{f(u)_J f(u)_{\hat{J}}} a_y + (\text{a linear combination of terms } a_w \text{ with } l(w) > l(y)\ ).$$

Proof: (1) Since $y \in Y_J$, $y = (yw_{o\hat{J}})w_{o\hat{J}}$ with $yw_{o\hat{J}} \in X_{\hat{J}}$ and $l(y)=l(yw_{o\hat{J}})+l(w_{o\hat{J}})$. Then for all $w \in W_{\hat{J}}$, $l(yw_{o\hat{J}}w)=l(yw_{o\hat{J}})+l(w)$

By (5.6(2)) we have

$$\begin{aligned} a_y o_{\hat{J}} &= (-1)^{l(w_{o\hat{J}})} a_{yw_{o\hat{J}}} o_{\hat{J}} \\ &= \frac{(-1)^{l(w_{o\hat{J}})}}{f(u)_{\hat{J}}} \sum_{w \in W_{\hat{J}}} (-1)^{l(w)} u^{c_{ww_{o\hat{J}}}} a_{yw_{o\hat{J}}w} \\ &= \frac{1}{f(u)_{\hat{J}}} a_y + (\text{a combination of terms } a_w,\ l(w) < l(y)). \end{aligned}$$

So $a_y o_{\hat{J}} e_J = \dfrac{(-1)^{l(w_{o\hat{J}})}}{f(u)_{\hat{J}} f(u)_J} \displaystyle\sum_{w \in W_{\hat{J}}} \sum_{v \in W_J} (-1)^{l(w)} u^{c_{ww_{o\hat{J}}}} a_{yw_{o\hat{J}}w} a_v.$

Since $l(yw_{oJ}) = l(y)+l(w_{oJ})$, $a_y a_{w_{oJ}} = a_{yw_{oJ}}$. Now for all $w \in W_J$, $v \in W_J$, $a_{yw_{o\hat{J}}w} a_v = \sum k_{w''} a_{w''}$, summed over certain $w'' \in W$ with $l(w'') \leqslant l(yw_{o\hat{J}}w)+l(v) = l(yw_{o\hat{J}})+l(w)+l(v)$

$$\leqslant l(yw_{o\hat{J}})+l(w_{o\hat{J}})+l(w_{oJ}).$$

Hence $a_{yw_{oJ}}$ can only occur with non-zero coefficient in a product $a_{yw_{o\hat{J}}w}a_v$ where $l(w)=l(w_{o\hat{J}})$ and $l(v)=l(w_{oJ})$, that is, where $w=w_{o\hat{J}}$ and $v=w_{oJ}$. Now if $w\neq w_{o\hat{J}}$, for all $v \in W_J$,

$a_{yw_{o\hat{J}}w}a_v = \Sigma\, k_{w''}a_{w''}$, summed over certain $w'' \in W$ with $l(w'') \leqslant l(yw_{o\hat{J}})+l(w)+l(v) < l(yw_{oJ})$. Hence

$$a_y o_{\hat{J}} e_J = \frac{1}{f(u)_{\hat{J}} f(u)_J} a_{yw_{oJ}} + (\text{a linear combination of terms } a_w \text{ with } l(w) < l(yw_{oJ})\ ).$$

(2) Since $y \in Y_J$, then for all $w \in W_J$, $l(yw) = l(y)+l(w)$ and $a_y a_w = a_{yw}$. Then

$$a_y e_J o_{\hat{J}} = \frac{1}{f(u)_J f(u)_{\hat{J}}} \sum_{\substack{w \in W_J \\ w\neq 1}} \sum_{v \in W_{\hat{J}}} (-1)^{l(v)} u^{c_{vw_{o\hat{J}}}} a_{yw}a_v + \frac{1}{f(u)_J} a_y o_{\hat{J}}.$$

Now if $w\neq 1$, $w \in W_J$, $a_{yw}a_v = \Sigma\, k_{w''}a_{w''}$ summed over certain $w'' \in W$ with $l(yw)-l(v) \leqslant l(w'')$ (since $v \in W_{\hat{J}}$, $l(y) \geqslant l(v)$). As $l(yw) > l(y)$ and $l(v) \leqslant l(w_{o\hat{J}})$, $l(yw_{o\hat{J}})=l(y)-l(w_{o\hat{J}})$, $l(yw_{o\hat{J}}) < l(yw)-l(v)$ for any $v \in W_{\hat{J}}$. From (1) we have

$$a_y o_{\hat{J}} = \frac{1}{f(u)_{\hat{J}}} a_{yw_{o\hat{J}}} + (\text{a combination of terms } a_w,\ l(w) > l(yw_{o\hat{J}})).$$

Hence $$a_y e_J o_{\hat{J}} = \frac{1}{f(u)_J f(u)_{\hat{J}}} a_{yw_{o\hat{J}}} + (\text{a linear combination of certain } a_w,\ l(w) > l(yw_{o\hat{J}})).$$

(5.12) <u>THEOREM</u>: (1) The elements $\{a_y o_{\hat{J}} e_J : y \in Y_J\}$ are linearly independent, for all $J \subseteq R$.

(2) The elements $\{a_y e_J o_{\hat{J}} : y \in Y_J\}$ are linearly independent, for all $J \subseteq R$.

Proof: (1) Suppose we have a relation $\sum_{y \in Y_J} b_y a_y o_{\hat{J}} e_J = 0$,

where the $b_y \in B_o$. Set $S_n = \sum_{\substack{y \in Y_J \\ l(y) \leq n}} b_y a_y o_{\hat{J}} e_J$. Then if $S_n = 0$,

by (5.11(1)). $b_y = 0$ for all $y \in Y_J$ with $l(y)=n$,

as $\{a_w : w \in W\}$ are a basis of A, and so $S_{n-1}=0$.

Let m be the length of the maximal element of Y_J. Then $S_m=0$ is the given relation. By decreasing induction it follows that $b_y = 0$ for all $y \in Y_J$. Hence the given elements are linearly independent.

(2) Suppose we have a relation $\sum_{y \in Y_J} b_y a_y e_J o_{\hat{J}} = 0$,

where the $b_y \in B_o$. Set $S_n = \sum_{\substack{y \in Y_J \\ l(y) \geq n}} b_y a_y e_J o_{\hat{J}}$. Then if $S_n = 0$,

by (5.11(2)) $b_y = 0$ for all $y \in Y_J$ with $l(y)=n$, as $\{a_w : w \in W\}$ are a basis of A, and so $S_{n+1}=0$. The relation given above is $S_m = 0$, where m is the length of the shortest element in Y_J. Thus by induction, all $b_y=0$, and so the given elements are linearly independent.

(5.13) THEOREM: (1) The elements $\{a_y o_{\hat{J}} e_J : y \in Y_J,\ J \subseteq R\}$ are linearly independent and hence form a basis of A.

(2) The elements $\{a_y e_J o_{\hat{J}} : y \in Y_J,\ J \subseteq R\}$ are linearly independent and hence form a basis of A.

Proof: (1) Suppose we have a relation $\sum_{J \subseteq R} \sum_{y \in Y_J} b_y a_y o_{\hat{J}} e_J = 0$

where the $b_y \in B_o$. We may suppose there exists at least one non-zero coefficient b_y such that $b_y = {}^{g(u)}/h(u)$ with $u \nmid g(u)$ and $u \nmid h(u)$, $g(u)$, $h(u) \in Q[u]$.

Consider the specialisation f_o of B_o defined in (3.4.11). f_o induces a ring epimorphism $f_o' : A \to H_Q$, the 0-Hecke algebra of type (W,R) over Q. In particular, writing the standard basis elements of H_Q as $\{h_w : w \in W\}$ and the idempotents defined in (4.4.1) as E_J and O_J respectively, we see that $f_o'(a_y o_{\hat{J}} e_J) = h_y O_{\hat{J}} E_J$. Now apply f_o' to the given relation:

$$\sum_{J \subsetneq R} \sum_{y \in Y_J} f_o(b_y) h_y O_{\hat{J}} E_J = 0.$$

As $\{h_y O_{\hat{J}} E_J : y \in Y_J,\ J \subsetneq R\}$ are linearly independent, $f_o(b_y) = 0$ for all $y \in W$. But $\ker f_o = uB_o$, and we have supposed that at least one of the non-zero coefficients b_y was not in uB_o. Hence all b_y are zero, and the given elements are linearly independent.

(Note: If all $b_y \in uB_o$, then the original relation becomes $u \sum_{J \subsetneq R} \sum_{y \in Y_J} b_y' a_y o_{\hat{J}} e_J = 0$ for some $b_y' \in B_o$, and as B_o is an integral domain and A an algebra, we have $\sum_{J \subsetneq R} \sum_{y \in Y_J} b_y' a_y o_{\hat{J}} e_J = 0$.)

(2) Use a similar argument.

We would like to express A as a direct sum of left ideals of the form $Ao_{\hat{J}} e_J$ or $Ae_J o_{\hat{J}}$. Notice that for all

$y \in Y_J$, $a_y o_{\hat{J}} e_J \in A o_{\hat{J}} e_J$ and $a_y e_J o_{\hat{J}} \in A e_J o_{\hat{J}}$, and so $\dim A o_{\hat{J}} e_J \geq |Y_J|$ and $\dim A e_J o_{\hat{J}} \geq |Y_J|$. If we can show that $\dim A o_{\hat{J}} e_J = |Y_J|$ and $\dim A e_J o_{\hat{J}} = |Y_J|$ for all $J \subseteq R$, then $\{a_y o_{\hat{J}} e_J : y \in Y_J\}$ is a basis of $A o_{\hat{J}} e_J$ and $\{a_y e_J o_{\hat{J}} : y \in Y_J\}$ is a basis of $A e_J o_{\hat{J}}$; then by (5.13) we have $A = \sum^{\oplus}_{J \subseteq R} A o_{\hat{J}} e_J$ and $A = \sum^{\oplus}_{J \subseteq R} A e_J o_{\hat{J}}$. We begin by examining Ae_J and Ao_J for all $J \subseteq R$.

(5.14) <u>THEOREM</u>: $Ae_J = \{a \in A : ao_{\{w_j\}} = 0 \text{ for all } w_j \in J\}$

$= \{a \in A : ae_{\{w_j\}} = a \text{ for all } w_j \in J\}$.

Ae_J has basis $\{a_y e_J : y \in X_J\}$ and dimension $|X_J|$. Further, $\{a_y o_{\hat{L}} e_L : y \in Y_L,\ J \subseteq L\}$ is a basis of Ae_J and $Ae_J = \sum_{J \subseteq L} A o_{\hat{L}} e_L$.

<u>Proof</u>: Clearly, $Ae_J \leq \{a \in A : ao_{\{w_j\}} = 0 \text{ for all } w_j \in J\}$. Conversely, suppose $a \in A$ satisfies $ao_{\{w_j\}} = 0$ for all $w_j \in J$. Then $aa_{w_j} = u^{c_j} a$ for all $w_j \in J$, and so $ae_J = \frac{1}{f(u)_J} \sum_{w \in W_J} aa_w$. But $aa_w = u^{c_w} a$ for all $w \in W_J$, and so $ae_J = a$. Then $a \in Ae_J$, and $Ae_J = \{a \in A : ao_{\{w_j\}} = 0 \text{ for all } w_j \in J\}$.

Clearly also, $Ae_J \leq \{a \in A : ae_{\{w_j\}} = a \text{ for all } w_j \in J\}$. $ae_{\{w_j\}} = a$ gives $a(1 + a_{w_j}) = (u^{c_j} + 1)a$, and thus $aa_{w_j} = u^{c_j} a$, so as before, $Ae_J = \{a \in A : ae_{\{w_j\}} = a \text{ for all } w_j \in J\}$.

Let $a = \sum_{w \in W} b_w a_w \in Ae_J$. Let $w_j \in J$. Then $ao_{\{w_j\}} = 0$ gives $\sum_{w \in W} b_w a_w (u^{c_j} 1 - a_{w_j}) = 0$. So

$$\sum_{w \in W} u^{c_j} b_w a_w - \sum_{\substack{w \in W \\ w(r_j) > 0}} b_w a_{ww_j} - \sum_{\substack{w \in W \\ w(r_j) < 0}} b_w (u^{c_j} a_{ww_j} + (u^{c_j}-1)a_w) = 0.$$

Since $\{a_w : w \in W\}$ is a basis of A, the coefficient of each a_w is zero. Suppose $w(r_j) > 0$; the coefficient of a_w is

$$u^{c_j} b_w - u^{c_j} b_{ww_j} .$$

As this is zero, $b_w = b_{ww_j}$. Similarly if $w(r_j) < 0$, the coefficient of a_w gives us that $b_w = b_{ww_j}$.

Let $x \in X_J$ and $w \in W_J$. Then writing $w = w_{i_1} .. w_{i_t}$, $l(w)=t$, $w_{i_j} \in J$ for all j, we see that

$$b_x = b_{xw_{i_1}} = b_{xw_{i_1}w_{i_2}} = .. = b_{xw}.$$

Then $a = \sum_{x \in X_J} f(u)_J b_x a_x e_J$. Conversely, each $a_x e_J \in Ae_J$, and as $\{a_x e_J : x \in X_J\}$ is linearly independent, it is a basis of Ae_J. Thus $\dim Ae_J = |X_J|$.

Finally, for all $L \supseteq J$ we have $Ao_L^{\wedge} e_L \leqslant Ae_J$, and so $\sum_{J \subseteq L} Ao_L^{\wedge} e_L \leqslant Ae_J$. Moreover, $\{a_y o_L^{\wedge} e_L : y \in Y_L, L \supseteq J\}$ is a set of linearly independent elements in $\sum_{J \subseteq L} Ao_L^{\wedge} e_L$, so also in Ae_J. Since $\dim Ae_J = |X_J|$, this set is also a basis of Ae_J, and so we must have $Ae_J = \sum_{J \subseteq L} Ao_L^{\wedge} e_L$.

(5.15) <u>THEOREM</u>: $Ao_J = \{a \in A : ae_{\{w_j\}} = 0 \text{ for all } w_j \in J\}$

$= \{a \in A : ao_{\{w_j\}} = a \text{ for all } w_j \in J\}$

Ao_J has basis $\{a_y o_J : y \in X_J\}$ and dimension $|X_J|$. Further $\{a_y e_L^{\wedge} o_L : y \in Y_L, L \supseteq J\}$ is a basis of Ao_J and $Ao_J = \sum_{J \subseteq L} Ae_L^{\wedge} o_L$.

Proof: Clearly $Ao_J \leqslant \{a \in A: ae_{\{w_j\}}=0 \text{ for all } w_j \in J\}$. Conversely, suppose $a \in A$ satisfies $ae_{\{w_j\}}=0$ for all $w_j \in J$. Then $aa_{w_j} = -a$ for all $w_j \in J$, and so $aa_w = (-1)^{l(w)}a$ for all $w \in W_J$. Then $ao_J = a$, and so $a \in Ao_J$. Thus

$Ao_J = \{a \in A: ae_{\{w_j\}}=0 \text{ for all } w_j \in J\}$.

Similarly we can show that

$Ao_J = \{a \in A: ao_{\{w_j\}}=a \text{ for all } w_j \in J\}$.

Now let $a = \sum_{w \in W} b_w a_w \in Ao_J$. Let $w_j \in J$; then $ae_{\{w_j\}}=0$ gives $\sum_{w \in W} b_w a_w(1 + a_{w_j}) = 0$. So

$$\sum_{w \in W} b_w a_w + \sum_{\substack{w \in W \\ w(r_j) > 0}} b_w a_{ww_j} + \sum_{\substack{w \in W \\ w(r_j) < 0}} b_w(u^{c_j}a_{ww_j} + (u^{c_j}-1)a_w) = 0.$$

Since $\{a_w : w \in W\}$ is a basis of A, the coefficient of each a_w in the above expression is zero. Suppose $w(r_j) > 0$; the coefficient of a_w is $b_w + u^{c_j}b_{ww_j} = 0$. Thus $b_w = -u^{c_j}b_{ww_j}$, when $w(r_j) > 0$. If $w(r_j) < 0$, the coefficient of a_w is $b_w + b_{ww_j} + b_w(u^{c_j}-1) = b_{ww_j} + u^{c_j}b_w = 0$.

Let $x \in X_J$ and $w \in W_J$, with $w = w_{i_1}\dots w_{i_t}$, $l(w)=t$, $w_{i_j} \in J$, and then:

$$b_x = -u^{c_{i_1}}b_{xw_{i_1}} = u^{c_{i_1}+c_{i_2}}b_{xw_{i_1}w_{i_2}} = \dots = (-1)^{l(w)}u^{c_w}b_{xw}.$$

Hence $a = \sum_{x \in X_J} f(u)_J b_x a_x o_J$, and conversely each $a_x o_J \in Ao_J$ for all $x \in X_J$. As $\{a_x o_J : x \in X_J\}$ are linearly independent, they are a basis of Ao_J, and $\dim Ao_J = |X_J|$.

Finally, for all $L \supseteq J$ we have $Ae_{\hat{L}}o_L \leqslant Ao_J$, and so $\sum_{J \subseteq L} Ae_{\hat{L}}o_L \leqslant Ao_J$. Moreover, $\{a_y e_{\hat{L}} o_L : y \in Y_{\hat{L}},\ L \supseteq J\}$

is a set of linearly independent elements in $\sum_{J \subseteq L} Ae_{\hat{L}}o_L$,

so also in Ao_J. Since $\dim Ao_J = |X_J|$, this set must

also be a basis of Ao_J, and hence $Ao_J = \sum_{J \subseteq L} Ae_{\hat{L}}o_L$.

(5.16) <u>THEOREM</u>: (1) For all $J \subseteq R$, $Ae_J = \sum^{\oplus}_{J \subseteq L} Ao_{\hat{L}}e_L$.

(2) For all $J \subseteq R$, $Ao_J = \sum^{\oplus}_{J \subseteq L} Ae_{\hat{L}}o_L$.

<u>Proof</u>: Let $a \in A$, then $ao_{\hat{J}}e_J \in Ao_{\hat{J}}e_J \leqslant Ae_J$. Since

$\{a_y o_{\hat{L}} e_L : y \in Y_L,\ L \supseteq J\}$ forms a basis of Ae_J, there exist

elements $b_y \in B_o$ such that

$$ao_{\hat{J}}e_J = \sum_{J \subseteq L} \sum_{y \in Y_L} b_y a_y o_{\hat{L}} e_L.$$

Then $ao_{\hat{J}}e_J o_{\hat{J}} = \sum_{J \subseteq L} \sum_{y \in Y_L} b_y a_y o_{\hat{L}} e_L o_{\hat{J}} = \sum_{y \in Y_J} b_y a_y o_{\hat{J}} e_J o_{\hat{J}}$

since $L \cap \hat{J} \neq \emptyset$ if $J \subset L$. Hence $\dim Ao_{\hat{J}}e_J o_{\hat{J}} \leqslant |Y_J|$.

By a similar argument, $\dim Ae_{\hat{J}}o_J e_{\hat{J}} \leqslant |Y_{\hat{J}}|$.

Now $Ae_J = \sum_{J \subseteq L} Ao_{\hat{L}}e_L$, and so $Ae_J o_{\hat{J}} = (\sum_{J \subseteq L} Ao_{\hat{L}}e_L)o_{\hat{J}}$.

Thus $Ae_J o_{\hat{J}} \leqslant \sum_{J \subseteq L} Ao_{\hat{L}}e_L o_{\hat{J}} = Ao_{\hat{J}}e_J o_{\hat{J}}$. Since B_o is a

principal ideal domain,

$$\dim Ae_J o_{\hat{J}} \leqslant \dim Ao_{\hat{J}}e_J o_{\hat{J}} \leqslant |Y_J|.$$

But we have previously shown $\dim Ae_J o_{\hat{J}} \geqslant |Y_J|$, and so

$\dim Ae_J o_{\hat{J}} = |Y_J|$. Hence $\{a_y e_J o_{\hat{J}} : y \in Y_J\}$ is a basis of $Ae_J o_{\hat{J}}$.

Similarly, $\{a_y o_{\hat{J}} e_J : y \in Y_J\}$ is a basis of $Ao_{\hat{J}}e_J$. Thus

$Ae_J = \sum^{\oplus}_{J \subseteq L} Ao_{\hat{L}}e_L$, and $Ao_J = \sum^{\oplus}_{J \subseteq L} Ae_{\hat{L}}o_L$.

(5.17) COROLLARY: $Ao_{\hat{J}}e_J \cong Ae_J / \sum_{J \subset L} Ae_L$ and

$Ae_{\hat{J}}o_J \cong Ao_J / \sum_{J \subset L} Ao_L$ as left A-modules for all $J \subseteq R$.

Proof: As $Ae_J = \sum^{\oplus}_{J \subseteq L} Ao_{\hat{L}}e_L$, define the left A-module homomorphism $f: Ae_J \to Ao_{\hat{J}}e_J$ by projection. Clearly f is onto, and $\ker f = \sum^{\oplus}_{J \subset L} Ao_{\hat{L}}e_L = \sum_{J \subset L} Ae_L$. Also as $Ao_J = \sum^{\oplus}_{J \subseteq L} Ae_{\hat{L}}o_L$, we can define a left A-module homomorphism $g: Ao_J \to Ae_{\hat{J}}o_J$ by projection, and so the result follows.

(5.18) COROLLARY: (1) $A = \sum^{\oplus}_{J \subseteq R} Ao_{\hat{J}}e_J$ where $Ao_{\hat{J}}e_J$ has basis $\{a_y o_{\hat{J}} e_J : y \in Y_J\}$ and dimension $|Y_J|$.

(2) $A = \sum^{\oplus}_{J \subseteq R} Ae_{\hat{J}}o_J$ where $Ae_{\hat{J}}o_J$ has basis $\{a_y e_{\hat{J}} o_J : y \in Y_{\hat{J}}\}$ and dimension $|Y_{\hat{J}}|$.

Proof: Follows from (5.16) as in its proof we got that $\dim Ao_{\hat{J}}e_J = |Y_J|$ and $\dim Ae_{\hat{J}}o_J = |Y_{\hat{J}}|$.

Note: Let K be any extension ring of B_0, and let $A_K = A_K(u)$ be the generic ring over K. Then $A_K \cong K \otimes_{B_0} A$, and we also have the decomposition of (5.18) for A_K. In particular, if $K = Q(u)$, then we have the decompositions of (5.18) of A_K as a direct sum of 2^n left ideals, $n = |R|$.

Note also that the left ideals of A_K for any extension of B_0 which occur in the decompositions given in (5.18) are not necessarily indecomposable left ideals.

(5.19) COROLLARY: For all $J \subseteq R$, $Ao_{\hat{J}}e_J$ and $Ae_Jo_{\hat{J}}$ are isomorphic A-modules.

Proof: By (5.16), $Ao_{\hat{J}}e_Jo_{\hat{J}} = Ae_Jo_{\hat{J}}$ for all $J \subseteq R$ and $Ao_{\hat{J}}e_Jo_{\hat{J}}$ has basis $\{a_yo_{\hat{J}}e_Jo_{\hat{J}}: y \in Y_J\}$. Consider the A-module homomorphism $\Psi_J: Ao_{\hat{J}}e_J \to Ao_{\hat{J}}e_Jo_{\hat{J}}$ given by right multiplication by $o_{\hat{J}}$. If $\sum_{y \in Y_J} u_ya_yo_{\hat{J}}e_Jo_{\hat{J}} \in Ao_{\hat{J}}e_Jo_{\hat{J}}$, then $\Psi_J: \sum_{y \in Y_J} u_ya_yo_{\hat{J}}e_J \mapsto \sum_{y \in Y_J} u_ya_yo_{\hat{J}}e_Jo_{\hat{J}}$, and so Ψ_J is onto. Moreover, as $\Psi_J(a_yo_{\hat{J}}e_J) = a_yo_{\hat{J}}e_Jo_{\hat{J}}$ for all $y \in Y_J$, and the $\{a_yo_{\hat{J}}e_Jo_{\hat{J}}: y \in Y_J\}$ are a basis of $Ao_{\hat{J}}e_Jo_{\hat{J}}$, it follows that Ψ_J is one-one. Hence Ψ_J is an isomorphism of A-modules, and $Ao_{\hat{J}}e_J \cong Ao_{\hat{J}}e_Jo_{\hat{J}} = Ae_Jo_{\hat{J}}$ for all $J \subseteq R$.

(5.20) REMARK:

(1) Suppose $y \in Y_{\hat{J}}$; then $y = (yw_{oJ})w_{oJ}$ with $yw_{oJ} \in X_J$ and $l(y) = l(yw_{oJ})+l(w_{oJ})$. Now

$$a_yo_{\hat{J}}e_J = \frac{1}{f(u)_{\hat{J}}}u^{c_{w_{oJ}}}a_ye_J + \frac{1}{f(u)_{\hat{J}}}\sum_{\substack{w \in W_{\hat{J}}\\ w\neq 1}}(-1)^{l(w)}u^{c_{ww_{o\hat{J}}}}a_{yw}e_J.$$

Then $a_ye_J = a_{yw_{oJ}}e_Ju^{c_{w_{oJ}}}$ by (5.6)

$$= \frac{1}{f(u)_J}\sum_{v \in W_J}u^{c_{w_{oJ}}}a_{yw_{oJ}v} \quad \text{where for all}$$

$v \in W_J$, $l(yw_{oJ}v) = l(yw_{oJ})+l(v)$.

Also, for any $w \in W_{\hat{J}}$, $w\neq 1$, $a_{yw}e_J = \sum_{w'}k_{w'}a_{w'}$ where $k_{w'} \in B_o$, summed over certain $w' \in W$ with

$$l(w') \geqslant l(y)+1-l(w_{oJ}) > l(yw_{oJ}).$$

Hence $a_yo_{\hat{J}}e_J = \frac{1}{f(u)_{\hat{J}}f(u)_J}u^{c_{w_{o\hat{J}}}}u^{c_{w_{oJ}}}a_{yw_{oJ}}$ + (a sum of certain terms a_w with $l(w) > l(yw_{oJ})$).

Thus it follows that $\{a_y o_{\hat{J}} e_J : y \in Y_{\hat{J}}\}$ are a set of linearly independent elements in $Ao_{\hat{J}}e_J$, and so are a basis of $Ao_{\hat{J}}e_J$ since $|Y_{\hat{J}}| = |Y_J|$ for all $J \subseteq R$.

(2) Similarly, if $y \in Y_{\hat{J}}$, then $a_y e_J o_{\hat{J}} = a_{yw_{oJ}} u^{c_{w_{oJ}}} e_J o_{\hat{J}}$, and so $a_y e_J o_{\hat{J}} = \dfrac{1}{f(u)_J f(u)_{\hat{J}}} u^{c_{w_{oJ}}} (-1)^{l(w_{o\hat{J}})} a_{yw_{o\hat{J}}}$ + (a linear

combination of certain terms a_w, with

$l(w) < l(yw_{o\hat{J}}) = l(y)+l(w_{o\hat{J}})$).

Thus $\{a_y e_J o_{\hat{J}} : y \in Y_{\hat{J}}\}$ are also a basis of $Ae_J o_{\hat{J}}$ for all $J \subseteq R$.

(5A) Some Specialisations of $K = Q(u)$.

For any extension ring K' of $\sigma = Q[u]$, let $A_{K'} = A_{K'}(u)$ be the generic ring of the system S of finite groups with (B,N) pairs of type (W,R) over K', with identity $a_1 = 1$ and basis $\{a_w : w \in W\}$ as in (3.4.2). Let $K = Q(u)$. Define the idempotents e_J, o_J for each $J \subseteq R$ as in (5.5), and then by (5.18) and the note after it, we have the following two decompositions of A_K into direct sums of left ideals:

$$A_K = \sum_{J \subseteq R}^{\oplus} A_K o_{\hat{J}} e_J \qquad \text{and} \qquad A_K = \sum_{J \subseteq R}^{\oplus} A_K e_J o_{\hat{J}}$$

where $A_K o_{\hat{J}} e_J$ has dimension $|Y_J|$ and basis $\{a_y o_{\hat{J}} e_J : y \in Y_J\}$ for all $J \subseteq R$, and $A_K e_J o_{\hat{J}}$ has dimension $|Y_J|$ and basis $\{a_y e_J o_{\hat{J}} : y \in Y_J\}$ for all $J \subseteq R$.

Now for all $J \subseteq R$, and for all $y \in Y_J$, the element $a_y o_{\hat{J}} e_J$ has the form $\frac{1}{f(u)_{\hat{J}} f(u)_J}$ (an element of A_σ) and the element $a_y e_J o_{\hat{J}}$ has the form $\frac{1}{f(u)_J f(u)_{\hat{J}}}$ (an element of A_σ). We say $a_y o_{\hat{J}} e_J$ and $a_y e_J o_{\hat{J}}$ are defined in $A_{K'}$, for any extension ring K' of σ if $\frac{1}{f(u)_J}$ and $\frac{1}{f(u)_{\hat{J}}}$ are both elements of K'. If K' is an extension ring of σ which is contained in K such that $\{a_y o_{\hat{J}} e_J : y \in Y_J,\ J \subseteq R\}$ are all defined in $A_{K'}$, then they are linearly independent over K', and $A_{K'}$ has a decomposition $A_{K'} = \sum_{J \subseteq R}^{\oplus} A_{K'} o_{\hat{J}} e_J$, where $A_{K'} o_{\hat{J}} e_J$ has dimension $|Y_J|$ and basis $\{a_y o_{\hat{J}} e_J : y \in Y_J\}$. Note that $\{a_y o_{\hat{J}} e_J : y \in Y_J,\ J \subseteq R\}$ are all defined in $A_{K'}$ if and only if $\{a_y e_J o_J^{\wedge} : y \in Y_J,\ J \subseteq R\}$ are all defined in $A_{K'}$. Hence $A_{K'}$ also has the decomposition $A_{K'} = \sum^{\oplus}_{J \subseteq R} A_{K'} e_J o_J^{\wedge}$ where $A_{K'} e_J o_J^{\wedge}$ has dimension $|Y_J|$ and basis $\{a_y e_J o_J^{\wedge} : y \in Y_J\}$ for all $J \subseteq R$.

Definition: Suppose $\mathcal{O} \leqslant K' \leqslant K$, and $\{a_y o_J^{\wedge} e_J : y \in Y_J,\ J \subseteq R\}$ are defined in $A_{K'}$. Then we say $A_{K'}$ has the Solomon decomposition property, SDP.

Let P be a prime ideal of $\mathcal{O}$, and let $K_P = \{{}^a/b : a \in \mathcal{O},\ b \in \mathcal{O}-P\}$. Then K_P is a subring of K which contains $\mathcal{O}$. A specialisation F of K with nucleus P is a ring homomorphism $F: K_P \to C$ with kernel PK_P and image k_o, a subfield of C. F induces a ring epimorphism $F: A_{K_P} \to A_{k_o}(F(u))$. (see (3.4.4)).

Now let F be a specialisation of K with nucleus P and image k_o. Suppose $\dfrac{1}{f(u)_J} \in K_P$ for all $J \subseteq R$. Then A_{K_P} has the SDP, and we have the decompositions

$$A_{K_P} = \sum^{\oplus}_{J \subseteq R} A_{K_P} o_J^{\wedge} e_J \quad \text{and} \quad A_{K_P} = \sum^{\oplus}_{J \subseteq R} A_{K_P} e_J o_J^{\wedge}$$

where $A_{K_P} o_J^{\wedge} e_J$ has dimension $|Y_J|$ and basis $\{a_y o_J^{\wedge} e_J : y \in Y_J\}$ and $A_{K_P} e_J o_J^{\wedge}$ has dimension $|Y_J|$ and basis $\{a_y e_J o_J^{\wedge} : y \in Y_J\}$ for all $J \subseteq R$. Now F induces a ring epimorphism $F: A_{K_P} \to A_{k_o}(c)$, where $F(u) = c \in C$. Let $F(e_J) = E_J$ and $F(o_J) = O_J$. E_J and O_J are idempotents as F is a ring epimorphism. Further, F induces module epimorphisms F_J and

and F_J' as follows:

$$F_J : A_{K_P} o_{\hat{J}} e_J \to A_{k_o}(c) O_{\hat{J}} E_J \text{ defined by } F_J(a o_{\hat{J}} e_J) = F(a) O_{\hat{J}} E_J$$

and $F_J' : A_{K_P} e_J o_{\hat{J}} \to A_{k_o}(c) E_J O_{\hat{J}}$ defined by $F_J'(a e_J o_{\hat{J}}) = F(a) E_J O_{\hat{J}}$

where $a \in A_{K_P}$ for all $J \subseteq R$. Since F is an epimorphism $F : A_{K_P} \to A_{k_o}(c)$, it follows that

$$A_{k_o}(c) = \sum_{J \subseteq R}^{\oplus} A_{k_o}(c) O_{\hat{J}} E_J \quad \text{and} \quad A_{k_o}(c) = \sum_{J \subseteq R}^{\oplus} A_{k_o}(c) E_J O_{\hat{J}}$$

where $A_{k_o}(c) O_{\hat{J}} E_J$ and $A_{k_o}(c) E_J O_{\hat{J}}$ are left ideals of $A_{k_o}(c)$ of dimension $|Y_J|$, with bases $\{a_y O_{\hat{J}} E_J : y \in Y_J\}$ and $\{a_y E_J O_{\hat{J}} : y \in Y_J\}$ respectively.

<u>EXAMPLES</u>:

(1) Let $P = u\sigma$ and let $F : K_P \to C$ be the Q-linear map defined by $F(u) = 0$. Then the image of F is Q, and F induces an epimorphism $F : A_{K_P} \to A_Q(0) \cong H_Q$, the 0-Hecke algebra of type (W,R) over Q. If E_J and O_J are the idempotents defined in (4.4.1), then we see that $F(e_J) = E_J$ and $F(o_J) = O_J$ for all $J \subseteq R$, and the resulting decompositions of H_Q which we obtain are the same as obtained in (4.4.12).

(2) Let $P = (u-1)\sigma$ and let $F : K_P \to C$ be the Q-linear map defined by $F(u) = 1$. Then the image of F is Q, and F induces an epimorphism $F : A_{K_P} \to A_Q(1) \cong QW$. For all $J \subseteq R$,

$$F(e_J) = \frac{1}{|W_J|} \sum_{w \in W_J} w \quad \text{and} \quad F(o_J) = \frac{1}{|W_J|} \sum_{w \in W_J} (-1)^{l(w)} w,$$

which are the idempotents defined in (1.4.1). Then, letting

$A = QW$, we have the decompositions

$$A = \sum_{J \subseteq R}^{\oplus} AO_{\hat{J}}E_J \quad \text{and} \quad A = \sum_{J \subseteq R}^{\oplus} AE_JO_{\hat{J}}$$

where $AO_{\hat{J}}E_J$ and $AE_JO_{\hat{J}}$ are left ideals of A, of dimension $|Y_J|$ and bases $\{yO_{\hat{J}}E_J : y \in Y_J\}$ and $\{yE_JO_{\hat{J}} : y \in Y_J\}$ respectively. Note that the second decomposition is the one given in (1.4.2).

(3) For any $q \in \mathcal{P}$, let $P = (u-q)\mathcal{O}$. Let $F:K_P \to C$ be the Q-linear map defined by $F(u) = q$. Then the image of F is Q, and F induces a ring epimorphism $F:A_{K_P} \to A_Q(q) \cong H_Q(q)$. The idempotents E_J and O_J in $H_Q(q)$ are as follows:

$$E_J = \frac{1}{|G_J(q):B(q)|} \sum_{w \in W_J} h_w$$

$$O_J = \frac{1}{|G_J(q):B(q)|} \sum_{w \in W_J} (-1)^{l(w)} q^{l(w_{0J}w)} h_w$$

We thus have the decompositions $H_Q(q) = \sum_{J \subseteq R}^{\oplus} H_Q(q)O_{\hat{J}}E_J$ and $H_Q(q) = \sum_{J \subseteq R}^{\oplus} H_Q(q)E_JO_{\hat{J}}$, where for all $J \subseteq R$, $H_Q(q)O_{\hat{J}}E_J$ and $H_Q(q)E_JO_{\hat{J}}$ are left ideals of $H_Q(q)$ of dimension $|Y_J|$ and bases $\{h_yO_{\hat{J}}E_J : y \in Y_J\}$ and $\{h_yE_JO_{\hat{J}} : y \in Y_J\}$ respectively.

(5B) Decomposition Numbers of H_o.

The algebra H_o defined in (3.5) is not semi-simple, and $\{U_J = H_o o_J^{\wedge} e_J : J \subseteq R\}$ is a full set of the isomorphism classes of principal indecomposable H_o-modules. Also, $\{M_J = {}^{U}J/\mathrm{rad}\ U_J : J \subseteq R\}$ is a full set of isomorphism classes of the irreducible H_o-modules. The Cartan matrix C of H_o is defined in (4.5), and if $C = (c_{JL})$, then c_{JL} = the number of times M_L occurs as a composition factor of U_J.

Notation: $\sigma = Q[u]$

$K_o = Q(u)$

K = a finite field extension of K_o which is a splitting field for $A_K = A_K(u)$

I = integral closure of σ in K

k_o = subfield of C

$A_B = A_B(u)$ is the generic ring of the system S of finite groups with (B,N) pairs of type (W,R) over B, where B is any extension ring of σ.

f_o = a specialisation of K with nucleus P and range k_o such that $f_o(u) = 0$; f_o induces a ring epimorphism $f_o' : A_{K_P} \to A_{k_o}(0) = H_o$

(5B.1) THEOREM (Dornhoff [12], no.48.1(iv)): If M is any finitely generated A_K-module, then $M \cong X_K$ for some finitely generated K_P-free A_{K_P}-module X, where $X_K = K \otimes_{K_P} X$.

Proof: Since I is a Dedekind domain, K_P is a principal ideal domain. Let M have K-basis $m_1, .. m_n$ and let $\{a_w : w \in W\}$ be a K_P-basis of A_{K_P}. Then $\{1_K \otimes a_w : w \in W\}$ is a K-basis of $A_K = K \otimes_{K_P} A_{K_P}$. Let X be the A_{K_P}-submodule of M generated by all $(1_K \otimes a_w)m_i$. Then $K \otimes_{K_P} X = M$. Since M is K_P-torsion free, so is X; K_P is a principal ideal domain, and so X is K_P-free.

(5B.2) Definition: Let $\{X_i\}_{i=1}^{s}$ be a set of K_P-free A_{K_P}-modules such that $\{(X_i)_K\}$ are a set of irreducible A_K-modules.

(5B.3) Definition: The decomposition matrix of H_o, $D = (d_{iJ})$, is an $s \times 2^n$ matrix (where $n = |R|$) with entries d_{iJ}, $1 \leq i \leq s$, $J \subseteq R$, defined by:

d_{iJ} = the number of times M_J occurs as a composition factor of $f_o'(X_i)$.

Since $P \cap \mathcal{O} = u\mathcal{O}$, we can define idempotents e_J and o_J in A_{K_P} as follows:

(5B.4)
$$e_J = \frac{1}{f(u)_J} \sum_{w \in W_J} a_w$$
$$o_J = \frac{1}{f(u)_J} \sum_{w \in W_J} (-1)^{l(w)} u^{c_{wwoJ}} a_w$$

with notation as in (5.5). For all $J \subseteq R$, define the A_{K_P}-module $(U_J)_{K_P} = A_{K_P} o_J e_J$

and the A_K-module $(U_J)_K = A_K \hat{e}_J$. We see that

$$(U_J)_K = K \otimes_{K_p} (U_J)_{K_p}.$$

(5B.5) LEMMA (Dornhoff [12], 46.1): Let A be any ring, e an idempotent in A, and V an A-module. Define

$$f: eV \to \mathrm{Hom}_A(Ae, V)$$

by $f(v)(ae) = aev$ for all $v \in eV$, $a \in A$. Then

(1) f is an isomorphism of additive groups.

(2) if A is an R-algebra for the commutative ring R, then f is an R-isomorphism.

Proof: Clearly f is a group homomorphism. If $f(v) = 0$ for some $v \in V$, then $0 = f(v)e = ev = v$ so f is one-one. If $h \in \mathrm{Hom}_A(Ae,V)$, $h(e) = h(e^2) = eh(e)$, so $h(e) \in eV$. Then $f(h(e)) = h$ so f is onto.

(5B.6) THEOREM (Dornhoff [12], 48.4): Let U_J be a principal indecomposable H_o-module, and V any finitely generated K_p-free A_{K_p}-module. Then

$$\begin{aligned}\dim_{K_p} \mathrm{Hom}_{A_{K_p}}((U_J)_{K_p}, V) &= \dim_{k_o} \mathrm{Hom}_{H_o}(U_J, f_o(V)) \\ &= \dim_K \mathrm{Hom}_{A_K}((U_J)_K, V_K)\end{aligned}$$

where $V_K = K \otimes_{K_p} V$, and if $v_1, \ldots v_m$ are a K_p-basis of V, then $f_o(V) = \{ \sum_{i=1}^{m} f_o(k_i)v_i : k_i \in K_p, \sum_{i=1}^{m} k_i v_i \in V\}$.

Proof: There exist idempotents $E_J \in H_o$, $E_{J,K_p} \in A_{K_p}$ and $E_{J,K} \in A_K$ such that $U_J = H_o E_J$, $(U_J)_{K_p} = A_{K_p} E_{J,K_p}$ and $(U_J)_K = A_K E_{J,K}$. Then $\mathrm{Hom}_{A_{K_p}}(A_{K_p} E_{J,K_p}, V) \cong E_{J,K_p} V$, a finitely generated K_P-free K_P-module. Let $d = \dim_{K_P} \mathrm{Hom}_{A_{K_P}}(A_{K_P}E_{J,K_P}, V)$.

Now $\mathrm{Hom}_{A_K}(A_K E_{J,K}, V_K) \cong \mathrm{Hom}_{A_K}(A_K(1_K \otimes E_{J,K_P}), V_K)$

$$\cong (1_K \otimes E_{J,K_P})V_K \cong K \otimes_{K_P} E_{J,K_P}V.$$

Since $E_{J,K_P}V$ is K_P-free, this last has dimension d.

Finally, $\mathrm{Hom}_{H_o}(H_o E_J, f_o(V)) \cong E_J f_o(V) \cong f_o(E_{J,K_P}V)$ and this last has dimension d over k_o.

(5B.7) <u>LEMMA</u> (Dornhoff [12], 48.5): Let A be any ring, P a projective A-module, and assume that

$$0 \to L \xrightarrow{a} M \xrightarrow{b} N \to 0$$

is an exact sequence of A-modules. Define

$a^*: \mathrm{Hom}_A(P,L) \to \mathrm{Hom}_A(P,M)$ by $a^*(f) = a.f$

$b^*: \mathrm{Hom}_A(P,M) \to \mathrm{Hom}_A(P,N)$ by $b^*(g) = b.g$

Then $0 \to \mathrm{Hom}_A(P,L) \xrightarrow{a^*} \mathrm{Hom}_A(P,M) \xrightarrow{b^*} \mathrm{Hom}_A(P,N) \to 0$ is an exact sequence of abelian groups.

(5B.8) <u>THEOREM</u> (Dornhoff [12], 48.6): Let $U_J = H_o E_J$ for some idempotent $E_J \in H_o$. Let V be a finitely generated H_o-module, and assume M_L occurs n_L times as a composition factor of V. Then $n_L = \dim_{k_o} E_L V$.

<u>Proof</u>: Since V is a finitely generated H_o-module, there exists a sequence of submodules

$$0 = V_{i_{n+1}} < V_{i_n} < .. < V_{i_1} < V_{i_o} = V$$

such that V_{i_j} is a H_o-module for each $j, 0 \leq j \leq n+1$, and $V_{i_j}/V_{i_{j+1}} \cong M_{i_j}$, an irreducible H_o-module. Then we have an exact sequence of H_o-modules

$$0 \to V_{i_{j+1}} \to V_{i_j} \to M_{i_j} \to 0.$$

Then for each $J \subseteq R$,

$$0 \to \mathrm{Hom}_{H_o}(U_J, V_{i_{j+1}}) \to \mathrm{Hom}_{H_o}(U_J, V_{i_j}) \to \mathrm{Hom}_{H_o}(U_J, M_{i_j}) \to 0$$

is an exact sequence of abelian groups. In particular,

$$\dim_{k_o}\mathrm{Hom}_{H_o}(U_J, V_{i_j}) - \dim_{k_o}\mathrm{Hom}_{H_o}(U_J, V_{i_{j+1}})$$

$$= \dim_{k_o}\mathrm{Hom}_{H_o}(U_J, M_{i_j}).$$

So $\dim_{k_o}\mathrm{Hom}_{H_o}(U_J, V) = \sum_j \dim_{k_o}\mathrm{Hom}_{H_o}(U_J, M_{i_j})$

$$= \sum_L n_L \dim_{k_o}\mathrm{Hom}_{H_o}(U_J, M_L).$$

Now if $J \neq L$, $\mathrm{Hom}_{H_o}(U_J, M_L) = 0$, and if $J = L$, then $\dim_{k_o}\mathrm{Hom}_{H_o}(U_J, M_J) = 1$. Hence

$$\dim_{k_o}\mathrm{Hom}_{H_o}(U_J, V) = \dim_{k_o} E_J V = n_J.$$

(5B.9) LEMMA (Dornhoff [12], 48.8(i)): $(U_J)_K = \sum_i d_{iJ}(X_i)_K$

Proof: Since A_K is semi-simple,

$$(U_J)_K = \sum_{i=1} a_{iJ}(X_i)_K, \text{ where the } a_{iJ} \in Z.$$

Since K is a splitting field for A_K,

$$\mathrm{Hom}_{A_K}((X_i)_K, (X_j)_K) = \begin{cases} K \text{ if } i=j \\ (0) \text{ if } i \neq j \end{cases}$$

and so $a_{iJ} = \dim_K \mathrm{Hom}_{A_K}((U_J)_K, (X_i)_K)$

$= \dim_{k_o}\mathrm{Hom}_{H_o}(U_J, f_o(X_i))$ by (5B.6)

$= \dim_{k_o} E_J f_o(X_i) = d_{iJ}.$

(5B.10) THEOREM (Dornhoff [12], 48.8(ii)): $C = D^tD$, where D^t is the transpose of D.

Proof: By definition of c_{JL}, $c_{JL} = \dim_{k_o} E_L U_J$. Thus

$$\begin{aligned} c_{JL} &= \dim_{k_o} \mathrm{Hom}_{H_o}(U_L, U_J) \text{ by (5B.5)} \\ &= \dim_K \mathrm{Hom}_{A_K}((U_L)_K, (U_J)_K) \text{ by (5B.6)} \\ &= \dim_K \mathrm{Hom}_{A_K}\left(\sum_i d_{iL}(X_i)_K, \sum_j d_{jJ}(X_j)_K\right) \text{ by (5B.9)} \\ &= \sum_i d_{iL} d_{iJ}. \end{aligned}$$

EXAMPLE: Let H_o be of type $(W(A_3), \{w_1, w_2, w_3\})$. Then from Starkey [22], appendix 6, we have that the decomposition matrix D of H_o is as follows:

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Then } D^tD = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which is the Cartan matrix of H_o given in Appendix 4, no. (3).

Chapter 6: THE RELATIVE STEINBERG MODULES OF A FINITE GROUP WITH (B,N) PAIR.

Introduction:

Let G(q) be a Chevalley group over the field GF(q), where $q=p^n$ for some prime integer p, and $n > 0$. Steinberg [23] showed that G(q) has a remarkable irreducible character of degree equal to a power of q. Later, Curtis [8] showed that any finite group G with a (B,N) pair has an irreducible character μ which is equal to Steinberg's in the case $G = G(q)$, and μ may be written as an alternating linear combination of characters induced from characters of parabolic subgroups of G:

$$\mu = \sum_{J \leq R} (-1)^{|J|} (1_{G_J})^G$$

where 1_{G_J} is the principal character of the parabolic subgroup G_J which corresponds to the subset J of R. Solomon [21] shows that this formula has a homological source when $|R| \geq 2$, i.e. that μ corresponds to the representation of G on a 'homology module' for a particular simplicial complex.

(6.1) The Relative Steinberg Modules of a Finite Group with a (B,N) Pair.

Definition: The relative Steinberg character of type J of a finite group G with a (B,N) pair, where $J \subseteq R$, is the character

$$\mu_J = \sum_{\substack{K \\ J \leqslant K \leqslant R}} (-1)^{|K-J|} (1_{G_K})^G$$

where 1_{G_K} is the principal character of the parabolic subgroup G_K of G.

When $|\hat{J}| \geqslant 2$, we will show that μ_J corresponds to the representation of G on a 'homology module'for a subcomplex of the complex considered by Solomon above.

<u>The Tits Complex of G</u> (refer to Tits [26])

Let G be a finite group with (B,N) pair of rank n, and let $G^1,..G^n$ be the maximal parabolic subgroups of G containing B. (If $R = \{w_1,..w_n\}$, then for each $i = 1,..n$, let $R_i = \{w_1,..\hat{w}_i,..w_n\}$, and set $G^i = G_{R_i}$ for all i.) Let V^i be the collection of cosets gG^i, for $g \in G$, and let $V = V^1 \cup V^2 \cup .. \cup V^n$. Then the Tits complex of G is a simplicial complex Δ of dimension n-1 which has V as its set of vertices. A collection S of vertices is a simplex of Δ if and only if $\bigcap_{v \in S} v$ is non-empty. If S and S' are collections of vertices which are simplexes, we say S is a face of S' if every vertex of S is a vertex of S'.

<u>The Tits Complex of G with respect to G_J.</u>

Let $G^1,..G^n$ be the maximal parabolic subgroups of G containing B, numbered so that $G^1,..G^r$ contain G_J, but none of $G^{r+1},..G^n$ contain G_J.(In this case, we have that $\hat{J} = \{w_1,..w_r\}$). Let $V_J = V^1 \cup V^2 \cup .. \cup V^r$. Then the Tits complex of G with respect to G_J is a simplicial complex Δ_J of dimension r-1 which has V_J as its set of vertices. A collection S of vertices of V_J is a simplex of Δ_J if and only if $\bigcap_{v\in S} v$ is non-empty. If S and S' are collections of vertices of V_J which are simplexes, then S is a face of S' if every vertex of S is a vertex of S'.

Note that if $J = \emptyset, \Delta_\emptyset = \Delta$. Moreover, for each $J \subseteq R$, Δ_J is a subcomplex of Δ. In particular, if $J,K \subsetneq R$, and $J \subseteq K$, then Δ_K is a subcomplex of Δ_J.

Let $S = \{g_{i_o}G^{i_o},..g_{i_p}G^{i_p}\}$, $1\leq i_o< i_1< ..<i_p\leq r$ be a collection of vertices of V_J which form a p-simplex σ of Δ_J. Write

$$\sigma = (g_{i_o}G^{i_o},..g_{i_p}G^{i_p}),\ 1\leq i_o< i_1< ..<i_p\leq r.$$

There is a natural G-action on Δ_J defined as follows: if $\sigma = (g_{i_o}G^{i_o},..g_{i_p}G^{i_p})$ is a p-simplex of Δ_J, $g \in G$, define the p-simplex $g\sigma = (gg_{i_o}G^{i_o},..gg_{i_p}G^{i_p})$ of Δ_J. $g\sigma$ is a p-simplex of Δ_J as $\bigcap_{j=0}^{p} gg_{i_j}G^{i_j} = g\bigcap_{j=0}^{p} g_{i_j}G^{i_j}$ is non-empty as σ is a p-simplex of Δ_J.

Note that we have defined an ordering on the simplexes of Δ_J by insisting that the vertices of a simplex be written in the order above.

For each subset $L = \{w_{i_o},..w_{i_p}\}$ of $\hat{J}$, define the standard p-simplex σ_L as follows:

$$\sigma_L = (G^{i_o},..G^{i_p}),\ \text{where } 1\leq i_o< i_1< ..<i_p\leq r.$$

(6.1.1) LEMMA: Each p-simplex σ of Δ_J is conjugate under G to precisely one σ_L, for some $L \subseteq \hat{J}$, $|L| = p+1$.

Proof: Let $\sigma = (g_{i_o}G^{i_o},..g_{i_p}G^{i_p})$, $1 \leq i_o < i_1 < .. < i_p \leq r$, be any p-simplex of Δ_J. Then there exists $g \in G$, $g \neq 0$, such that $g \in \bigcap_{j=0}^{p} g_{i_j}G^{i_j}$. Then $gG^{i_j} \cap g_{i_j}G^{i_j} \neq \emptyset$ for each j, and so $gG^{i_j} = g_{i_j}G^{i_j}$ for all j. So $\sigma = (gG^{i_o},..gG^{i_p}) = g(G^{i_o},..G^{i_p})$, i.e. $\sigma = g\sigma_L$, where $L = \{w_{i_o},..w_{i_p}\}$.

Now suppose $\sigma = g_1\sigma_{L_1} = g_2\sigma_{L_2}$, with $g_1, g_2 \in G$, and $L_1, L_2 \subset \hat{J}$. Then $\sigma_{L_1} = g\sigma_{L_2}$, where $g = g_1^{-1}g_2 \in G$. Thus, for each $w_i \in L_1$, there exists $w_{j(i)} \in L_2$ such that $G^i = gG^{j(i)}$. So $g \in G^{j(i)}$ as $1 \in G^i$; continuing in this way we get that $g \in \bigcap_{w_j \in L_2} G^j$, and hence $g\sigma_{L_2} = \sigma_{L_2} = \sigma_{L_1}$. Hence the result.

(6.1.2) LEMMA: (1) $\{g \in G\colon g\sigma_L = \sigma_L\} = G_{\hat{L}}$.

(2) If $\sigma = g\sigma_L$, then $\{x \in G\colon \sigma = x\sigma_L\} = gG_{\hat{L}}$.

Proof: (1) $g\sigma_L = \sigma_L$ if and only if $g \in \bigcap_{w_i \in L} G^i = G_{\hat{L}}$.

(2) Let $\sigma = g\sigma_L$. Then $g\sigma_L = x\sigma_L$ if and only if $g^{-1}x\sigma_L = \sigma_L$. By (1), $\sigma_L = g^{-1}x\sigma_L$ if and only if $g^{-1}x \in G_{\hat{L}}$, i.e. $g\sigma_L = x\sigma_L$ if and only if $x \in gG_{\hat{L}}$.

(6.1.3) PROPOSITION: There is a one-one correspondence between simplexes σ of Δ_J and cosets gG_K of G with $J \subseteq K \subset R$ and $g \in G$, given by

$$\sigma = g(G^{i_o},..G^{i_p}) \to g(\bigcap_{j=0}^{p} G^{i_j}) = gG_K, \text{ where } \hat{K} = \{w_{i_o},..w_{i_p}\}.$$

<u>Definition</u>: A simplex of dimension r-1 in Δ_J is called a chamber. The chamber $\sigma_{\hat{J}} = (G^1, .. G^r)$ is called the fundamental chamber.

In the remainder of this section, we will prove the following two main theorems:

(6.1.4) <u>THEOREM 1</u>: Let Δ_J be the Tits complex of a finite group G with a (B,N) pair with respect to a parabolic subgroup G_J, with $|\hat{J}| = r \geq 2$. Then the homology groups of Δ_J with integral coefficients are:

$$H_0(\Delta_J) \cong Z$$

$$H_i(\Delta_J) = 0 \text{ for any } i \text{ with } 1 \leq i \leq r-2, \text{ or } i \geq r.$$

$$H_{r-1}(\Delta_J) \cong Z \oplus Z \oplus .. \oplus Z \text{ (t summands)}$$

where $t = \sum_{w \in Y_J} |B : B \cap B^{w^{-1}}|$. If ϵ: $W \to \{\pm 1\}$ is the alternating character of W and $\sigma_{\hat{J}}$ is the fundamental chamber of Δ_J, then the (r-1)-chains

$$\left.\begin{array}{l} z_1 = \sum_{w \in W_{\hat{J}}} \epsilon(w) y_1 w \sigma_{\hat{J}} \\ \vdots \\ z_l = \sum_{w \in W_{\hat{J}}} \epsilon(w) y_l w \sigma_{\hat{J}} \end{array}\right\} \text{ where } Y_J = \{y_1, .. y_l\}$$

are cycles. If $\left.\begin{array}{l} b_{11}, .. b_{1i_1} \\ \vdots \\ b_{l1}, .. b_{li_l} \end{array}\right\}$ are coset representatives for

$\left.\begin{array}{l} B \cap B^{y_1^{-1}} \\ \vdots \\ B \cap B^{y_l^{-1}} \end{array}\right\}$ in B, then $\left.\begin{array}{l} b_{11} z_1, .. b_{1i_1} z_1 \\ \vdots \\ b_{l1} z_l, .. b_{li_l} z_l \end{array}\right\}$ are cycles which form a basis for $H_{r-1}(\Delta_J)$.

Note: If $G = G(q)$ is a finite Chevalley group, $t \doteq \sum_{w \in Y_J} q^{l(w)}$.

(6.1.5) THEOREM 2: Let Δ_J be the Tits complex of a finite group G with a (B,N) pair with respect to a parabolic subgroup G_J, with $|\hat{J}| \geq 2$. Then the action of G on Δ_J defines a $Q[G]$-module structure on $H_{r-1}(\Delta_J) \otimes Q$ which affords the character

$$\mu_J = \sum_{\substack{K \\ J \leq K \leq R}} (-1)^{|K-J|} (1_{G_K})^G.$$

We call the G-module $H_{r-1}(\Delta_J) \otimes Q$ the relative Steinberg module of type J.

(6.1.6) LEMMA: For each $K \subseteq R$,

$$G = \bigcup_{w \in X_K} BwG_K$$

where X_K is defined as in (1.3.1).

(6.1.7) COROLLARY: Each simplex σ of Δ_J has the form $bw\sigma_L$ for a unique subset L of $\hat{J}$ and a unique $w \in X_{\hat{L}}$.

Let D_J denote the set of chambers of Δ_J. We say $c_1, c_2 \in D_J$ are adjacent if $c_1 \cap c_2$ is a simplex of dimension r-2.

Definition: If $\sigma \in \Delta_J$, define $D_J(\sigma) = \{c \in D_J : \sigma < c\}$, where ' $<$ ' means 'is a face of'.

Definition: For $\sigma \in \Delta_J$, define $l(\sigma) = \min_w l(w)$, over all $w \in W$ for which σ can be written $\sigma = bw\sigma_L$, some $L \subseteq \hat{J}$.

It is easy to see that $l(\sigma) = l(w)$ where $\sigma = bw\sigma_L$, with $L \subseteq \hat{J}$ and $w \in X_{\hat{L}}$.

(6.1.8) PROPOSITION: Let $\sigma = bw\sigma_L \in \Delta_J$, with $w \in X_{\hat{L}}$, $b \in B$, $L \subsetneq \hat{J}$. Then $D_J(\sigma) = \{b'ww'\sigma_J : w' \in X_J^{\hat{L}},\ b'ww'G_J \subseteq bwBw'G_J\}$, where $X_J^{\hat{L}} = \{w \in W_{\hat{L}} : w(\prod_J) \subseteq \Phi^+\}$.

This follows because if $J \subseteq K$, if $w \in X_K$ and $b \in B$, then $bwG_K = \bigcup_{w' \in X_J^K} bwBw'G_J$.

(6.1.9) LEMMA (Solomon [21]): Let $\sigma \in \Delta = \Delta_\emptyset$. Then

(1) there exists a unique $c_o \in D(\sigma) = D_\emptyset(\sigma)$ such that $l(c_o) \leq l(c)$ for all $c \in D(\sigma)$.

(2) if $c \in D(\sigma)$ and $l(c) = l(c_o) + m$, then there exist $c_o, c_1, \ldots c_m = c \in D(\sigma)$ such that c_i, c_{i+1} are adjacent and $l(c_i) = l(c_o) + i$ for all i, $0 \leq i \leq m-1$.

Proof: (1) Suppose $\sigma = bw\sigma_K$, $w \in X_{\hat{K}}$, for some $K \subsetneq R$. Then $D(\sigma) = \{b'ww'\sigma_R ; w' \in W_{\hat{K}},\ b'ww'B \subseteq bwBw'B\}$ and if $w' \in W_{\hat{K}}$, $l(ww') = l(w) + l(w')$. Then $c_o = bw\sigma_R$ has the required properties.

(2) Suppose $c \in D(\sigma)$. Then $c = b'ww'\sigma_R$, for some $b' \in B$, $w' \in W_{\hat{K}}$, where $b'ww'B \subseteq bwBw'B$. Let $w' = w_{i_1} \ldots w_{i_s}$, $l(w') = s$, and $w_{i_j} \in R$ for each j. Since $w \in X_{\hat{K}}$, $w' \in W_{\hat{K}}$, we have that $l(ww') = l(w) + l(w')$, and so $l(c) = l(c_o) + l(w') = l(c_o) + s$. For $j = 1,2,\ldots s$, set $c_j = b'ww_{i_1} \ldots w_{i_j}\sigma_R$. Then each $c_j \in D(\sigma)$, and $l(c_j) = l(c_o) + j$. It remains to show c_j, c_{j+1} are adjacent for $j = 0,1,\ldots s-1$. Since $ww_{i_1} \ldots w_{i_j}(r_{i_{j+1}}) \in \Phi^+$,

set $K = R-\{w_{i_{j+1}}\}$, and then $b'ww_{i_1}..w_{i_j}\sigma_K$ is a face of dimension $n-2$ of both c_j and c_{j+1}. Hence the result.

(6.1.10) <u>PROPOSITION</u>: Let $s \in \Delta_J$. Then if $s = bw\sigma_L$ with $w \in X_{\hat{L}}$,

(1) there exists a unique $c_o \in D_J(s)$, $c_o = bw\sigma_{\hat{J}}$, $w \in X_J$, such that if $c \in D_J(s)$, $c = b'w'\sigma_{\hat{J}}$, $w' \in X_J$, $w' \neq w$, then $l(c_o) < l(c)$.

(2) if $c \in D_J(s)$, $c \neq c_o$, then there exists a sequence $c_o, c_1, ..c_m = c \in D_J(s)$ such that c_i, c_{i+1} are adjacent for each $i = 0,1,..m-1$ and $l(c_i) < l(c_{i+1})$ for each $i = 0,1,..m-1$.

<u>Proof</u>: (1) Since $s = bw\sigma_L$, $w \in X_{\hat{L}}$, $D_J(s) = \{b'ww'\sigma_{\hat{J}}: w' \in X_J^{\hat{L}}, b'ww'G_J \subseteq bwBw'G_J\}$. $c_o = bw\sigma_{\hat{J}}$ has the required properties, for if $w' \in X_J^{\hat{L}}$, $l(ww') = l(w) + l(w')$, since $w \in X_{\hat{L}}$.

(2) Regard s as an element of $\Delta_\emptyset$. By (6.1.9) there exists a unique $\bar{c}_o = bw\sigma_R \in D_\emptyset(s)$ such that if $\bar{c} \in D_\emptyset(s)$, $\bar{c} \neq \bar{c}_o$, then $l(\bar{c}_o) < l(\bar{c})$. For $c = b'ww'\sigma_{\hat{J}} \in D_J(s)$, where $b'ww'G_J \subseteq bwBw'G_J$ and $w' \in X_J^{\hat{L}}$, consider $\bar{c} = b'ww'\sigma_R \in D_\emptyset(s)$. By (6.1.9), if $l(w') = m$, there exists a sequence $\bar{c}_o, \bar{c}_1, ..\bar{c}_m = \bar{c}$ of chambers in $D_\emptyset(s)$ such that $\bar{c}_i, \bar{c}_{i+1}$ are adjacent and $l(\bar{c}_i) = l(\bar{c}_o) + i$ for all i, $0 \leq i \leq m-1$. If $w' = w_{i_1}..w_{i_m}$, $w_{i_j} \in R$, then $\bar{c}_j = b'ww_{i_1}..w_{i_j}\sigma_R$. Now define $c_o, c_1, ..c_m = c$ as follows: $c_o = bw\sigma_{\hat{J}}$

$$c_j = b'ww_{i_1}..w_{i_j}\sigma_{\hat{J}} = b'ww(j)\sigma_{\hat{J}}, \text{ where } ww(j) \in X_J,$$

$l(ww(j)) = l(w) + l(w(j))$, and $l(w(j)) \leq l(w_{i_1}..w_{i_j})$.

We show by induction on j, for $j > 0$, that

(a) c_{j-1}, c_j are either equal or adjacent.

(b) $c_j \in D_J(s)$

(c) if c_{j-1}, c_j are adjacent, then $l(c_{j-1}) < l(c_j)$.

Then by omitting repetitions we have the required sequence of chambers. Now $c_0 = bw\sigma_{\hat{J}} \in D_J(s)$. Consider $c_1 = b'ww_{i_1}\sigma_{\hat{J}}$. If $w_{i_1} \in W_J$, then $c_0 = c_1$. So suppose $w_{i_1} \notin W_J$. Then $w \in X_{J \cup \{w_{i_1}\}}$, so $ww_{i_1} \in X_J$. Hence $c_1 = b'ww_{i_1}\sigma_{\hat{J}} \in D_J(s)$, $l(c_0) < l(c_1)$. Moreover, c_0 and c_1 are adjacent as they both contain $bw\sigma_{\hat{J}-\{w_{i_1}\}}$.

Suppose that for all $k < j$, we have that c_{k-1}, c_k are both in $D_J(s)$ and are either equal or adjacent, and if they are adjacent, $l(c_{k-1}) < l(c_k)$. Now $c_{j-1} = b'ww(j-1)\sigma_{\hat{J}}$, for some $w(j-1) \in X_J^{\hat{L}}$, and $c_j = b'ww_{i_1}..w_{i_j}\sigma_{\hat{J}}$. If $w_{i_j} \in W_J$, then $c_j = c_{j-1}$. So suppose $w_{i_j} \notin W_J$. Then we can write $ww_{i_1}..w_{i_j} = ww(j)w_J$, where $w_J \in W_J$, $w(j) \in X_J^{\hat{L}}$, and $l(ww(j)w_J) = l(w) + l(w(j)) + l(w_J)$. Then $c_j = b'ww(j)\sigma_{\hat{J}} \in D_J(s)$ as $w(j) \in X_J^{\hat{L}}$, $b'ww(j)G_J \subseteq bwBw(j)G_J$. c_{j-1} and c_j are adjacent as $b'ww(j-1)\sigma_{\hat{J}-\{w_{i_j}\}} = b'ww_{i_1}..w_{i_{j-1}}\sigma_{\hat{J}-\{w_{i_j}\}}$ is an r-2 dimensional face of both. $l(c_{j-1}) < l(c_j)$ by (1.3.8).

Let $\Delta(J)$ be the set of $s \in \Delta_J$ such that $s < c$ for some $c \in D_J$, $c = bw\sigma_{\hat{J}}$, with $w \in X_J$, $w \notin Y_J$.

(6.1.11) <u>LEMMA</u>: $\Delta(J) = \Delta_J - \{c_1,..c_t\}$, where $c_1,..c_t$ are all the chambers of the form $bw\sigma_{\hat{J}}$, $b \in B$, $w \in Y_J$. Further, $\Delta(J)$ is a subcomplex of Δ_J.

Proof: Clearly $c_1,\dots c_t \notin \Delta(J)$. Suppose s is a proper face of, say, c_1. We may suppose s is of dimension r-2. Then $s = bw\sigma_{\hat{J}-\{w_i\}}$, for some $w_i \in \hat{J}$, with $w \in X_{J\cup\{w_i\}}$, $b \in B$. So $w \notin Y_J$. But $D_J(s) = \{b'ww'\sigma_{\hat{J}}: w' \in X_J^{J\cup\{w_i\}}, b'ww'G_J \subseteq bwBw'G_J\}$. In particular, $s < c = bw\sigma_{\hat{J}}$, $w \in X_{J\cup\{w_i\}}$, $w \notin Y_J$. So $s \in \Delta(J)$.

Definition: If $s \in \Delta_J$, let $\Phi(s)$ be the subcomplex consisting of s and its faces.

Definition: Let $\Delta(J)_k$, for $k \in Z^+$, be the subcomplex of $\Delta(J)$ consisting of all $s \in \Delta(J)$ such that there exists a chamber $c = bw\sigma_{\hat{J}} \in D_J$, $l(w) \leq k$, $w \in X_J$ with $s < c$.

(6.1.12) LEMMA: $\Delta(J)_0 = \Phi(\sigma_{\hat{J}})$.

Proof: Clearly $\Phi(\sigma_{\hat{J}}) \subseteq \Delta(J)_0$. So suppose $s \in \Delta(J)_0$. Then $s = bw\sigma_L$, for some $L \subseteq \hat{J}$, $w \in X_{\hat{L}}$. Now $D_J(s) = \{b'ww'\sigma_{\hat{J}}: w' \in X_J^{\hat{L}}, b'ww'G_J \subseteq bwBw'G_J\}$. Since $s \in \Delta(J)_0$, $\sigma_{\hat{J}} = b'ww'\sigma_{\hat{J}}$ for some $b' \in B$, $w' \in X_J^{\hat{L}}$. But $ww' \in X_J$, and $l(ww') = l(w) + l(w')$. Hence $w = 1$, $w' = 1$, and $s = \sigma_L$. Thus $s \in \Phi(\sigma_{\hat{J}})$.

(6.1.13) LEMMA: Let $c \in D(J)$ = the set of chambers of $\Delta(J)$, and suppose $c = bw\sigma_{\hat{J}}$, $w \in X_J$, $l(w) = k > 0$. Let $s_1,\dots s_r$ be the r-2 dimensional faces of c numbered such that $s_1,\dots s_p \in \Delta(J)_{k-1}$, $s_{p+1},\dots s_r \notin \Delta(J)_{k-1}$. Then $1 \leq p \leq r-1$, and $\Phi(c) \cap \Delta(J)_{k-1} = \bigcup_{i=1}^{p} \Phi(s_i)$.

Proof: (a) Suppose $l(ww_i) = l(w) + 1$, with $w_i \in \hat{J}$. Then

$w \in X_{J\cup\{w_i\}}$ and $bw\sigma_{\hat{J}-\{w_i\}}$ is a face of maximal dimension in c.

(b) Suppose $l(ww_i) = l(w) - 1$, with $w_i \in \hat{J}$. Then $w = w_1(i)w_2(i)$, with $w_1(i) \in X_{J\cup\{w_i\}}$, $w_2(i) \in X_J^{J\cup\{w_i\}}$ and $l(w) = l(w_1(i)) + l(w_2(i))$. Then $bw_1(i)\sigma_{\hat{J}-\{w_i\}}$ is a face of maximal dimension in $bw\sigma_{\hat{J}}$, and $l(w_1(i)) < l(w)$.

Now suppose $w \in Y_{\hat{L}}$, with $L \subsetneq \hat{J}$. Assume that the $w_i \in \hat{J}$ are numbered such that $w_1,..w_p$ are the reflections in L, and $w_{p+1},..w_r$ those in $\hat{J}-L$. Let $s_i = bw_1(i)\sigma_{\hat{J}-\{w_i\}}$ if $1 \leq i \leq p$, and $s_j = bw\sigma_{\hat{J}-\{w_j\}}$ for $p+1 \leq j \leq r$. Then $s_1,..s_p \in \Delta(J)_{k-1}$ but $s_{p+1},..s_r \notin \Delta(J)_{k-1}$. As $k > 0$, $1 \leq p \leq r-1$.

For any $i \leq p$, $s_i \in \Delta(J)_{k-1}$, so $\Phi(s_i) \subsetneq \Delta(J)_{k-1}$. Hence $\bigcup_{i=1}^{p} \Phi(s_i) \subsetneq \Phi(c) \cap \Delta(J)_{k-1}$. Conversely, take any $t \in \Phi(c) \cap \Delta(J)_{k-1}$, such that $t \notin \bigcup_{i=1}^{p} \Phi(s_i)$. Obviously, $t \neq c$ as $t \in \Delta(J)_{k-1}$, and t cannot equal any of $s_{p+1},..s_r$, as none of these are in $\Delta(J)_{k-1}$. Since $\Phi(c) = c \cup \bigcup_{i=1}^{r} \Phi(s_i)$, we must have that $t < s_{q_1}$, $t < s_{q_2}$,..$t < s_{q_u}$ for some s_{q_i}, with $p+1 \leq q_i \leq r$ for all i. As $c \in D_J(t)$, there exists a unique $c_o \in D_J(t)$ such that $l(c_o) \leq l(c')$ for all $c' \in D_J(t)$. Now $c_o \neq c$ as $t \in \Delta(J)_{k-1}$, and so $l(c_o) \leq k-1$. By (6.1.10) there exists a sequence of chambers $c_o, c_1,..c_m = c$ in $D_J(t)$ such that c_i, c_{i+1} are adjacent for all i, $0 \leq i \leq m-1$, and $l(c_i) < l(c_{i+1})$. But then $c_{m-1} \cap c_m$ is an r-2 dimensional simplex contained in c_{m-1} and in c_m, and so must contain t.

So for some j, $1 \leq j \leq u$, $c_{m-1} \cap c_m = s_{q_j}$. Then as $s_{q_j} < c_{m-1}$, and $l(c_{m-1}) < l(c_m) = k$, $s_{q_j} \in \Delta(J)_{k-1}$ - contradiction. Hence $\Phi(c) \cap \Delta(J)_{k-1} = \bigcup_{i=1}^{p} \Phi(s_i)$.

(6.1.14)<u>LEMMA</u>: Let K be an abstract simplicial complex. If $s_1,..s_m$ are simplexes of K with at least one common vertex, then $\bigcup_{i=1}^{m} \Phi(s_i)$ has the homology of a point. (See Hilton and Wylie [15] for a proof).

(6.1.15) <u>COROLLARY</u>: $\phi(c) \cap \Delta(J)_{k-1} = \bigcup_{i=1}^{p} \Phi(s_i)$ has the homology of a point.

<u>Proof</u>: Since $p < r$, the $\{s_i\}_{i=1}^{p}$ have at least one common vertex.

(6.1.16) <u>LEMMA</u>: Let K be a simplicial complex which is a union $K = L \cup L_1 \cup L_2 \cup \quad \cup L_n$ of subcomplexes. Suppose

(a) each L_i has the homology of a point

(b) each $L \cap L_i$ has the homology of a point

(c) $L_i \cap L_j \subseteq L$ when $i \neq j$.

Then K and L have isomorphic homology groups.

<u>Proof</u>: Use induction on n. True for n=0. Write $K = K_1 \cup L_n$, where $K_1 = L \cup L_1 \cup L_2 \cup .. \cup L_{n-1}$, and by induction K and K_1 have isomorphic homology groups. Also, $K_1 \cap L_n$ has the homology of a point. Consider the Mayer-Vietoris exact sequence (see [15]):

$$\ldots \to H_p(K_1 \cap L_n) \to H_p(K_1) \oplus H_p(L_n) \to H_p(K_1 \cup L_n) \to$$
$$H_{p-1}(K_1 \cap L_n) \to \ldots$$

For all $p \geq 1$, $H_p(K_1 \cap L_n) = 0$, $H_p(L_n) = 0$, and $H_o(K_1 \cap L_n) \cong Z$, $H_o(L_n) \cong Z$. Thus if $p > 1$ we have

$$0 \to H_p(K_1) \to H_p(K_1 \cup L_n) \to 0$$

i.e. $H_p(K_1) \cong H_p(K_1 \cup L_n)$

If $p=1$, we have

$$0 \to H_1(K_1) \to H_1(K_1 \cup L_n) \to Z \to H_o(K_1) \oplus Z \to H_o(K_1 \cup L_n) \to 0.$$

We must show: $H_1(K_1) \cong H_1(K_1 \cup L_n)$

and $H_o(K_1) \cong H_o(K_1 \cup L_n)$

But $H_o(K_1 \cup L_n) \cong Z \oplus Z \oplus \ldots \oplus Z$, r copies, where r is equal to the number of connected components of $K_1 \cup L_n$. Since $H_o(L_n) \cong Z$, L_n is connected, and as $H_o(K_1 \cap L_n) \cong Z$, there is a vertex v of K_1 such that v is connected to every vertex of L_n. Hence the number of connected components of $K_1 \cup L_n$ equals the number of connected components of K_1. That is, $H_o(K_1 \cup L_n) \cong H_o(K_1)$. Hence consider

$$0 \to H_1(K_1) \xrightarrow{\alpha} H_1(K_1 \cup L_n) \xrightarrow{\beta} Z \xrightarrow{\gamma} H_o(K_1) \oplus Z \xrightarrow{\delta} H_o(K_1).$$

ker $\alpha = 0$, so α is a monomorphism. δ is a projection, and so γ is an injection. Thus ker $\gamma = 0$, and im $\beta = 0$. So ker $\beta = H_1(K_1 \cup L_n) \cong$ im α . α is an isomorphism and so $H_1(K_1) \cong H_1(K_1 \cup L_n)$.

Now fix an integer $k > 0$. Let $a_1, \ldots a_m$ be all the chambers of $\Delta(J)$ for which $l(a_i) = k$. Let $K = \Delta(J)_k$, $L = \Delta(J)_{k-1}$ and $L_i = \Phi(a_i)$. Each $\Phi(a_i)$ has the homology of a point, each $\Phi(a_i) \cap \Delta(J)_{k-1}$ has the homology of a point, and $\Phi(a_i) \cap \Phi(a_j)$ for $i \neq j$ is in $\Delta(J)_{k-1}$. So by (6.1.16), $\Delta(J)_k$ and $\Delta(J)_{k-1}$ have isomorphic homology groups. Now $\Delta(J)_0 = \Phi(\sigma_{\hat{J}})$ has the homology of a point, and so it follows by induction that $\Delta(J)_k$ has the homology of a point for all non-negative integers k. But $\Delta(J) = \Delta(J)_{k_0} = \Delta(J)_k$ for all integers $k \geq k_0$, for some integer k_0, and so $\Delta(J)$ has the homology of a point.

<u>Definition</u>: Let K be an oriented simplicial complex. Let $C_p(K)$ be the group of p-chains (with coefficients in Z), $d: C_p(K) \to C_{p-1}(K)$ the boundary operator for all p, $Z_p(K)$ the group of p-cycles of K, $B_p(K)$ the group of p-boundaries of K, and $H_p(K) = Z_p(K)/B_p(K)$ the pth-homology group of K.

<u>Proof of THEOREM 1</u>:

Recall that $\Delta(J) = \Delta_J - \{c_1, \ldots c_t\}$, where $c_1, \ldots c_t$ are all the chambers of the form $bw\sigma_{\hat{J}}$, $b \in B$, $w \in Y_J$.

We have $C_p(\Delta(J)) = C_p(\Delta_J)$ for all $p=0,1,\ldots r-2$, and so $H_p(\Delta_J) = H_p(\Delta(J))$ for all $p=0,\ldots r-2$. Since $\Delta(J)$ has the homology of a point, $H_0(\Delta_J) \cong Z$, and $H_i(\Delta_J)=0$ for $1 \leq i \leq r-2$.

Now $C_{r-1}(\Delta_J) \cong C_{r-1}(\Delta(J)) \oplus Z \oplus \ldots \oplus Z$ (t copies of Z). Consider $d: C_{r-1}(\Delta(J)) \oplus \underbrace{Z \oplus \ldots \oplus Z}_{t} \to C_{r-2}(\Delta(J)) = C_{r-2}(\Delta_J)$ and $d': C_{r-1}(\Delta(J)) \to C_{r-2}(\Delta(J))$. Since $H_{r-2}(\Delta_J) = H_{r-2}(\Delta(J)) = 0$, im d = im d' $\cong Z_{r-2}(\Delta_J) = Z_{r-2}(\Delta(J))$. But $H_{r-1}(\Delta(J)) = 0$, so ker d' = 0.

Thus $\ker d \cong Z \oplus Z \oplus .. \oplus Z$ (t copies). Hence $H_{r-1}(\Delta_J) \cong Z \oplus Z \oplus .. \oplus Z$ (t copies). Further, we have $H_i(\Delta_J) = 0$ for all $i>r$.

Finally, let $y_1 = w_{o\hat{J}} \in Y_J$. Then each $y_i \in Y_J$ has the form $y_i = w(i)y_1$, with $l(y_i) = l(w(i)) + l(y_1)$, and so $z_i = w(i)z_1$. So it is sufficient to show that z_1 is an (r-1)-cycle. Consider first $d(\sigma_{\hat{J}})$:

$$d(\sigma_{\hat{J}}) = d((G^1,..G^r)) = \sum_{i=1}^{r} (-1)^{i+1}(G^1,..\hat{G^i},..G^r)$$
$$= \sum_{i=1}^{r} (-1)^{i+1} \sigma_{\hat{J}-\{w_i\}}.$$

Now $z_1 = \sum_{w\in W_{\hat{J}}} \in(w)w_{o\hat{J}}w\sigma_{\hat{J}} = \sum_{w\in W_{\hat{J}}} \in(w_{o\hat{J}})\in(w)w\sigma_{\hat{J}} = \in(w_{o\hat{J}})z$,

where $z = \sum_{w\in W_{\hat{J}}} \in(w)w\sigma_{\hat{J}}$. So it is sufficient to show that

z is an (r-1)-cycle. Now

$$d(z) = \sum_{w\in W_{\hat{J}}} \in(w)w \sum_{i=1}^{r} (-1)^{i+1}\sigma_{\hat{J}-\{w_i\}}$$
$$= \sum_{i=1}^{r} (-1)^{i+1} \sum_{w\in W_{\hat{J}}} \in(w)w\sigma_{\hat{J}-\{w_i\}}.$$

For each $w \in W_{\hat{J}}$, either $l(ww_i) > l(w)$, and $w \in X^{\hat{J}}_{\{w_i\}}$, or $l(ww_i) < l(w)$, and then w has the form $w = w'w_i$ with $l(w) = l(w') + 1$. So

$$\sum_{w\in W_{\hat{J}}} \in(w)w\sigma_{\hat{J}-\{w_i\}} = \sum_{\substack{w\in W_{\hat{J}} \\ l(ww_i)>l(w)}} \in(w)w\sigma_{\hat{J}-\{w_i\}} + \sum_{\substack{w\in W_{\hat{J}} \\ l(ww_i)<l(w)}} \in(w)w\sigma_{\hat{J}-\{w_i\}}$$

$$= \sum_{\substack{w\in W_{\hat{J}} \\ l(ww_i)>l(w)}} \in(w)w\sigma_{\hat{J}-\{w_i\}} - \sum_{\substack{w\in W_{\hat{J}} \\ l(ww_i)>l(w)}} \in(w)w\sigma_{\hat{J}-\{w_i\}} = 0.$$

Hence z is an (r-1)-cycle, and thus $z_1,..z_l$ are all (r-1)-cycles. Each z_i has an expression in which there is a unique simplex, $y_i\sigma_{\hat{J}}$, whose length is strictly greater than the length of any other simplex which occurs with non-zero coefficient in the expression for z_i, and hence it follows that the t distinct (r-1)-cycles given are linearly independent and so form a basis for $H_{r-1}(\Delta_J)$. Thus we have proved THEOREM 1.

(6.1.17) <u>Conditions for an (r-1)-chain to be a cycle</u>.

We have already that

$$d(\sigma_{\hat{J}}) = d((G^1,..G^r)) = \sum_{i=1}^{r} (-1)^{i+1}\sigma_{\hat{J}-\{w_i\}}.$$

Consider an arbitrary (r-1)-chain $c = \sum_{\substack{w\in X_J \\ b\in B}} a_{bw}bw\sigma_{\hat{J}}$, where $a_{bw} \in Z$. Then $d(c) = \sum_{\substack{w\in X_J \\ b\in B}} a_{bw}bw\sum_{i=1}^{r}(-1)^{i+1}\sigma_{\hat{J}-\{w_i\}}$. Now each $w \in X_J$ can be written uniquely in the form $w = w_1(i)w_2(i)$ with $w_1(i) \in X_{J\cup\{w_i\}}$, $w_2(i) \in X_J^{J\cup\{w_i\}}$, and $l(w) = l(w_1(i)) + l(w_2(i))$. Then

$$d(c) = \sum_{i=1}^{r}(-1)^{i+1}\sum_{\substack{w\in X_J \\ b\in B}} a_{bw}bw_1(i)\sigma_{\hat{J}-\{w_i\}},$$

and so $d(c) = 0$ if and only if the coefficient of each (r-2)-simplex $bw\sigma_{\hat{J}-\{w_i\}}$, $w \in X_{J\cup\{w_i\}}$, is zero. Hence $d(c) = 0$ if and only if for each (r-2)-simplex $bw\sigma_{\hat{J}-\{w_i\}}$, $w \in X_{J\cup\{w_i\}}$, $1\leqslant i\leqslant r$, we have

$$\sum_{\substack{w' \in X_J^{J\cup\{w_i\}} \\ b'ww'G_J \subsetneq bwBw'G_J}} a_{b'ww'} = 0.$$

Rewrite this as $a_{bw} + \sum_{\substack{1 \neq w' \in X_J^{J \cup \{w_i\}} \\ b'ww'G_J \subseteq bwBw'G_J}} a_{b'ww'} = 0$. So if $w \in Y_K$ for some $K \supsetneq J$, then a_{bw}, for any $b \in B$, can be expressed as a sum of $a_{b'w'}$'s, with $l(w') > l(w)$. Hence we conclude that each a_{bw} can be expressed as a linear combination of $\{a_{b'y}: y \in Y_J, b' \in B\}$.

Proof of THEOREM 2:

Each $g \in G$ induces a non-singular linear transformation of $C_p(\Delta_J) \otimes_Z Q$, $H_{r-1}(\Delta_J) \otimes_Z Q$, and $H_o(\Delta_J) \otimes_Z Q$, giving rise to characters ϕ_p, θ_{r-1} and θ_o respectively. By the Hopf trace formula, we have

$$(-1)^{r-1}\theta_{r-1} + \theta_o = \sum_{p=0}^{r-1} (-1)^p \phi_p.$$

G acts as a permutation representation on the oriented simplexes of $C_p(\Delta_J)$, and the orbits are subsets

$$O_L = \{g\sigma_L: g \in G, L \subseteq \hat{J}\}.$$

So ϕ_p is the character of the permutation representation of G on $C_p(\Delta_J) \otimes_Z Q$. Let x_L be the character of the permutation representation of G on O_L. Then $\phi_p = \sum_L x_L$, summed over all $L \subseteq \hat{J}$ with $|L| = p+1$. Hence

$$\sum_{p=0}^{r-1} (-1)^p \phi_p = -\sum_{\emptyset \neq L \subseteq \hat{J}} (-1)^{|L|} x_L.$$

Now θ_o is the character of the representation of G on $H_o(\Delta_J) \otimes_Z Q$. As Δ_J is connected, θ_o is the principal character of G, that is, $\theta_o = 1_G$.

G acts transitively on O_L. Let $U_L = \{u \in G : u\sigma_L = \sigma_L\}$. Then $u \in U_L$ if and only if $u \in G_{\hat{L}}$. So $U_L = G_{\hat{L}}$. x_L is the character of the permutation representation of G on O_L, and so corresponds to the character of the permutation representation of G on $G_{\hat{L}}$: that is, $x_L = (1_{G_{\hat{L}}})^G$. Thus

$$-\sum_{\emptyset \neq L \subseteq \hat{J}} (-1)^{|L|} x_L = -\sum_{\emptyset \neq L \subseteq \hat{J}} (-1)^{|L|} (1_{G_{\hat{L}}})^G$$

$$= -\sum_{\substack{K \\ J \leqslant K < R}} (-1)^{|R-K|} (1_{G_K})^G$$

$$= -\sum_{\substack{K \\ J \leqslant K < R}} (-1)^{|K-J|} (1_{G_K})^G (-1)^r$$

Hence $(-1)^{r-1}\theta_{r-1} = \sum_{\substack{K \\ J \leqslant K < R}} (-1)^{r-1}(-1)^{|K-J|}(1_{G_K})^G - 1_G$

That is, $\theta_{r-1} = \sum_{\substack{K \\ J \leqslant K \leqslant R}} (-1)^{|K-J|}(1_{G_K})^G$, and THEOREM 2 has been proved.

(6.1.18) <u>COROLLARY</u> : The restriction of μ_J to $Q[B]$ is given by $\mu_{J/B} = \sum_{w \in Y_J} (1_{B \cap B^{w^{-1}}})^B$

<u>Proof</u>: B permutes the basis elements of $H_{r-1}(\Delta_J) \otimes_Z Q$ in the same way as B permutes the cosets of $B \cap B^{w^{-1}}$ in B, for $w \in Y_J$.

(6.1.19) <u>COROLLARY</u>: The space of B-invariant vectors in $H_{r-1}(\Delta_J) \otimes_Z Q$ is of dimension $|Y_J|$.

(6.1.20) <u>COROLLARY</u>: Let K be any field. Let Δ_J be the Tits complex of a finite group G with (B,N) pair with respect to a parabolic subgroup G_J of G, with $|J| = r \geq 2$. Then regarding Z as a subring of K in a natural way, we have that $H_o(\Delta_J) \otimes_Z K \cong K$

$$H_i(\Delta_J) \otimes_Z K = 0 \text{ for all } i,\ 1 \leq i \leq r-2, \text{ and } i \geq r$$

$$H_{r-1}(\Delta_J) \otimes_Z K \cong K \oplus K \oplus \ldots \oplus K \text{ (t copies)}$$

where $t = \sum_{w \in Y_J} |B : B \cap B^w|$. Further, the action of G on Δ_J defines a $K[G]$-module structure on $H_{r-1}(\Delta_J) \otimes_Z K$ which affords the character

$$\mu_J = \sum_{\substack{L \\ J \leq L \leq R}} (-1)^{|L-J|} (1_{G_L})^G,$$

where 1_{G_L} is the principal character of G_L.

<u>Note</u>: We call any $K[G]$-module M which affords the character μ_J above, where K and G are as in (6.1.20), a relative Steinberg module of type J. Corollary (6.1.20) gives us one such module $M = H_{r-1}(\Delta_J) \otimes_Z K$ whenever $|J|=r \geq 2$.

(6.2) <u>A Degenerate Form of the Relative Steinberg Modules.</u>

The Coxeter complex C of a finite Coxeter system (W,R) can be viewed as a degenerate form of the Tits building of a finite group G with a (B,N) pair (G,B,N,R) whose Weyl group is W. The maximal parabolic subgroups $W^1,..W^n$ of W (where $n=|R|$) are used to define the simplicial complex C in the same way that the maximal parabolic subgroups $G^1,..G^n$ of G are used to define Δ. Similarly, we can define the Coxeter complex C_J of W with respect to W_J, for $J \subsetneq R$.

Using similar arguments to those for the relative Tits complex Δ_J of G with respect to G_J, we have the following results:

(6.2.1) <u>THEOREM 1</u>: Let C_J be the simplicial Coxeter complex of a finite Coxeter system (W,R) with respect to a parabolic subgroup W_J of W, with $|\hat{J}|=r\geq 2$. Then the homology groups of C_J with integral coefficients are as follows:

$$H_0(C_J) \cong Z$$

$$H_i(C_J) = 0, \text{ if } 1\leq i\leq r-2 \text{ or } i\geq r.$$

$$H_{r-1}(C_J) \cong Z \oplus Z \oplus..\oplus Z, \text{ t summands,}$$

where $t = |Y_J|$. If $\epsilon : W \to \{\pm 1\}$ is the alternating character of W, and $\sigma_{\hat{J}}$ is the fundamental chamber of C_J, then the (r-1)-chains $z_i = \sum_{w\in W_{\hat{J}}} \epsilon(w)y_i w\sigma_{\hat{J}}$, for $1\leq i\leq t$, where $Y_J = \{y_1,..y_t\}$, are cycles which form a basis for $H_{r-1}(C_J)$.

(6.2.2) THEOREM 2: Let C_J be the Coxeter complex of a finite Coxeter system (W,R) with respect to a parabolic subgroup W_J of W, with $|\hat{J}|=r\geqslant 2$. Then the action of W on C_J defines a $Q[W]$-module structure on $H_{r-1}(C_J)\otimes_Z Q$ which affords the character

$$\epsilon_J = \sum_{\substack{K \\ J\leqslant K\leqslant R}} (-1)^{|K-J|}(1_{W_K})^W$$

where 1_{W_K} is the principal character of W_K.

(6.2.3) COROLLARY: Let $A = Q[W]$, and let e_J and o_J be the idempotents defined in (1.4.1). Then there is an isomorphism of the A-modules $Ao_{\hat{J}}e_J$ and $H_{r-1}(C_J)\otimes Q$, where $|\hat{J}| = r \geqslant 2$, given by mapping the basis elements $\{y_i o_{\hat{J}} e_J : 1\leqslant i\leqslant t\}$ of $Ao_{\hat{J}}e_J$ to the basis elements $\{z_i : 1\leqslant i\leqslant t\}$ of $H_{r-1}(C_J)\otimes Q$, where $y_i o_{\hat{J}} e_J \rightarrow z_i$ for all i, $1\leqslant i\leqslant t$.

Chapter 7: ON THE DECOMPOSITION OF THE MODULE L.

(7.1) General Decompositions of L.

Let G be a finite group with a (B,N) pair (G,B,N,R) of rank n with Weyl group W. Let K be a field. Let M be the principal KB-module (that is, M affords the representation 1_B of B); then $KG \otimes_{KB} M$ is the KG-module M^G induced from M, and affords the representation $(1_B)^G$ of G. Let $^G/B$ denote the set of right cosets of B in G and let L be the set of functions $f: {}^G/B \to K$, as in (3.1), after (3.1.10). L becomes a left KG-module by defining for all $x \in G$ the function $xf \in L$, given by $(xf)(Bg) = f(Bgx)$ for all $Bg \in {}^G/B$.

(7.1.1) THEOREM: L is isomorphic, as left KG-module, to $KG \otimes_{KB} M$.

Let $H = \mathrm{En}_{KG}(L)$. Then by (3.1.10) we have that H is generated as K-algebra with identity a_1 by $\{a_{w_i} : w_i \in R\}$ subject to the relations:

$$a_{w_i}^{\ 2} = q_i a_1 + (q_i - 1)a_{w_i} \text{ for all } w_i \in R$$

where $q_i = |B : B \cap B^{w_i}|$ for all $w_i \in R$.

$$(a_{w_i} a_{w_j} a_{w_i} \ldots)_{n_{ij}} = (a_{w_j} a_{w_i} a_{w_j} \ldots)_{n_{ij}} \text{ for all}$$

w_i, $w_j \in R$, $i \neq j$, where n_{ij} = order of $w_i w_j$ in W.
For all $w \in W$, define $a_w = a_{w_{i_1}} \ldots a_{w_{i_s}}$, where $w = w_{i_1} \ldots w_{i_s}$ is a reduced expression for w. Then H is the K-algebra with identity a_1 and K-basis $\{a_w : w \in W\}$, with an associative multiplication given by

$$a_{w_i}a_w = \begin{cases} a_{w_iw} & \text{if } l(w_iw) = l(w)+1 \\ q_ia_{w_iw} + (q_i-1)a_w & \text{if } l(w_iw) = l(w)-1. \end{cases}$$

for all $w \in W$, $w_i \in R$.

Now let S be a system of finite groups with (B,N) pairs of type (W,R). Then for each $G=G(q) \in S$, where $q \in \mathcal{P}$, we have $q_i = |B(q):B(q) \cap B(q)^{w_i}| = q^{c_i}$ for all $w_i \in R$. There are two cases to consider:

(1) Suppose that the characteristic of K is not equal to p, where $q = p^s$ for some positive integer s. We will assume that K is a field of characteristic 0 in this case, and then $H \cong H_K(q)$, as defined in (3.3).

(2) Suppose that the characteristic of K is equal to p. Then $H \cong H_K(0)$, which is defined in (3.1).

In chapters 4 and 5 we have given some decompositions of $H_K(q)$ and $H_K(0)$, which are as follows:

(1) $H_K(q) = \sum^{\oplus}_{J \subseteq R} E_JO_J^{\wedge}H_K(q)$ and $H_K(q) = \sum^{\oplus}_{J \subseteq R} O_J^{\wedge}E_JH_K(q)$,

where for all $J \subseteq R$, E_J and O_J are defined as follows:

$$E_J = \frac{1}{f(q)_J} \sum_{w \in W_J} a_w$$

$$O_J = \frac{1}{f(q)_J} \sum_{w \in W_J} (-1)^{l(w)} q^{c_{ww_{oJ}}} a_w$$

where $f(q)_J = \sum_{w \in W_J} q^{c_w}$.

(2) $H_K(0) = \sum^{\oplus}_{J \subseteq R} e_Jo_J^{\wedge}H_K(0)$ and $H_K(0) = \sum^{\oplus}_{J \subseteq R} o_J^{\wedge}e_JH_K(0)$,

where for all $J \subseteq R$, e_J and o_J are defined by:

$$e_J = \sum_{w \in W_J} a_w = \left[1 + a_{w_{oJ}}\right], \text{ with notation as in (4.4),}$$

$$o_J = (-1)^{l(w_{oJ})} a_{w_{oJ}}.$$

Further, for all $J \subseteq R$, $e_J o_{\hat{J}} H_K(0)$ and $o_{\hat{J}} e_J H_K(0)$ are indecomposable right ideals of $H_K(0)$.

(7.1.2) <u>THEOREM</u>: Let V be a module over a ring R. Then a direct sum decomposition of V, $V = \sum^{\oplus}_{i \in I} V_i$, is equivalent to writing 1_V as a sum of mutually orthogonal idempotents e_i, $1_V = \sum_{i \in I} e_i$, in $\mathrm{En}_R(V)$.

<u>Proof</u>: Suppose $V = \sum^{\oplus}_{i \in I} V_i$, a direct sum of R-submodules of V. Then for each $v \in V$, $v = \sum_{i \in I} v_i$ uniquely for some $v_i \in V_i$. Consider the map $e_i : V \to V$ given by $e_i(\sum_{j \in I} v_j) = v_i$ e_i is an R-endomorphism of V, and $e_i^2 = e_i \neq 0$ if $V_i \neq 0$. If $i \neq j$, then $e_i e_j = 0$. Finally, $1_V = \sum_{i \in I} e_i$.

Conversely, suppose that $1_V = \sum_{i \in I} e_i$, where the e_i are mutually orthogonal idempotents. Define $V_i = e_i V$ for each $i \in I$. Then for all $v \in V$, $v = 1_V v = \sum_{i \in I} e_i v$. Hence $V = \sum_{i \in I} V_i$. To show that this sum is direct, we have to show that the expression $v = \sum_{i \in I} e_i v$ is unique. Suppose $v = \sum_{i \in I} e_i v = \sum_{i \in I} v_i'$ where $v_i' \in V_i$ for all i. Then each v_i' is of the form $e_i v(i)$ for some $v(i) \in V_i$. Now for all $j \in I$, $e_j v = e_j(\sum_{i \in I} e_i v) = e_j(\sum_{i \in I} e_i v(i)) = e_j v(j) = v_j'$.

Hence $V = \sum^{\oplus}_{i \in I} V_i$.

Denote by $L(q)$ and $L(0)$ the KG-module L when K is a field of characteristic 0 or characteristic p respectively. Then we have

$$H_K(q) \cong En_{KG}(L(q))$$

and $$H_K(0) \cong En_{KG}(L(0)).$$

In the following work, if we do not wish to consider the characteristic of K, we will just write L.

(7.1.3) <u>COROLLARY</u>: (1) $L(q) = \sum^{\oplus}_{J \subseteq R} E_J O_{\hat{J}} L(q)$ and $L(q) = \sum^{\oplus}_{J \subseteq R} O_{\hat{J}} E_J L(q)$ are decompositions of $L(q)$ as direct sums of KG-submodules.

(2) $L(0) = \sum^{\oplus}_{J \subseteq R} e_J o_{\hat{J}} L(0)$ and $L(0) = \sum^{\oplus}_{J \subseteq R} o_{\hat{J}} e_J L(0)$ are decompositions of $L(0)$ as direct sums of indecomposable KG-submodules.

<u>Proof</u>: The decompositions of $H_K(q)$ and $H_K(0)$ correspond to expressing the identity of $H_K(q)$ and the identity of $H_K(0)$ as sums of mutually orthogonal idempotents: if, say, in $H_K(0)$ we have $1 = \sum_{J \subseteq R} p_J$, where $p_J \in o_{\hat{J}} e_J H_K(0)$, and the elements p_J for all $J \subseteq R$ are a set of mutually orthogonal primitive idempotents with $p_J H_K(0) = o_{\hat{J}} e_J H_K(0)$, then by (7.1.2) we have that $L(0) = \sum^{\oplus}_{J \subseteq R} p_J L(0)$. Since for all $J \subseteq R$ we have that $p_J L(0) = p_J H_K(0) L(0) = o_{\hat{J}} e_J H_K(0) L(0) = o_{\hat{J}} e_J L(0)$, the required decomposition follows. Similarly

in the other cases.

(7.1.4) Definition: Let f_{Bg} be the function of L defined by:

$$f_{Bg}(Bg') = \begin{cases} 0 \text{ if } Bg' \neq Bg \\ 1 \text{ if } Bg' = Bg \end{cases}$$

Then $\{f_{Bg}: Bg \in {}^G/B\}$ is a K-basis of L, and is called the set of characteristic functions of L.

(7.1.5) LEMMA: f_B generates the KG-module L.

Proof: Any element of L is a K-linear combination of elements f_{Bg}, for some $Bg \in {}^G/B$. Since for all $g \in G$ we have $f_{Bg} = g^{-1}f_B$, f_B generates L as KG-module.

(7.1.6) COROLLARY: (1) $E_J O_{\hat{J}} f_B$ and $O_{\hat{J}} E_J f_B$ generate the KG-modules $E_J O_{\hat{J}} L(q)$ and $O_{\hat{J}} E_J L(q)$ respectively.

(2) $e_J o_{\hat{J}} f_B$ and $o_{\hat{J}} e_J f_B$ generate the KG-modules $e_J o_{\hat{J}} L(0)$ and $o_{\hat{J}} e_J L(0)$ respectively.

(7.1.7) PROPOSITION: (1) $E_J L(q) = \{f \in L(q): O_{\{w_i\}} f = 0$ for all $w_i \in J\} = \{f \in L(q): E_{\{w_i\}} f = f$ for all $w_i \in J\}$. $E_J L(q)$ has dimension $\sum_{w \in X_J} q^{c_w} = |G(q):G_J(q)|$ and basis

$\{E_J f_{Bwb}: w^{-1} \in X_J, Bwb \subseteq BwB\}$. Further, let M_J be the principal $KG_J(q)$-module. Then $E_J L(q)$ and the KG-module $M_J{}^G$ are isomorphic KG-modules.

(2) $e_J L(0) = \{f \in L(0): o_{\{w_i\}} f = 0$ for all $w_i \in J\} = \{f \in L(0): e_{\{w_i\}} f = f$ for all $w_i \in J\}$.

$e_J L(0)$ has dimension $\sum_{w \in X_J} q^{c_w} = |G(q):G_J(q)|$ and basis

$\{e_J f_{Bwb}: w^{-1} \in X_J,\ Bwb \subseteq BwB\}$. Further, let M_J be the principal $KG_J(q)$-module. Then $e_J L(0)$ and the KG-module $M_J^{\ G}$ are isomorphic KG-modules.

<u>Proof</u>: (1) Clearly $E_J L(q) \leq \{f \in L(q): 0_{\{w_i\}} f = 0$ for all $w_i \in J\}$. Conversely, let $f \in L(q)$ satisfy $0_{\{w_i\}} f = 0$ for all $w_i \in J$. Then $(q^{c_i} 1 - a_{w_i}) f = 0$ for all $w_i \in J$; that is, $a_{w_i} f = q^{c_i} f$ for all $w_i \in J$. Hence for all $w \in W_J$, $a_w f = q^{c_w} f$. Then $E_J f = \frac{1}{f(q)_J} \sum_{w \in W_J} a_w f = \frac{1}{f(q)_J} \sum_{w \in W_J} q^{c_w} f = f$, and so $f \in E_J L(q)$.

Similarly, if $E_{\{w_i\}} f = f$, then $(1 + a_{w_i}) f = (q^{c_i} + 1) f$, and hence $a_{w_i} f = q^{c_i} f$, and it follows that

$E_J L(q) = \{f \in L(q): E_{\{w_i\}} f = f$ for all $w_i \in J\}$.

Since $\{f_{Bwb}: w \in W,\ Bwb \subseteq BwB\}$ form a K-basis of $L(q)$, let $f = \sum_{w \in W} \sum_{Bwb \subseteq BwB} k_{Bwb} f_{Bwb} \in E_J L(q)$, where the $k_{Bwb} \in K$.

Let $w_i \in J$; then $0_{\{w_i\}} f = 0$ gives

$$* \quad q^{c_i} \sum_{w \in W} \sum_{Bwb \subseteq BwB} k_{Bwb} f_{Bwb} - \sum_{w \in W} \sum_{Bwb \subseteq BwB} k_{Bwb} a_{w_i} f_{Bwb} = 0.$$

Now for all $w \in W$, $a_{w_i} f_{Bwb} = \sum_{Bg \subseteq Bw_i Bwb} f_{Bg}$. Suppose $w^{-1}(r_i) > 0$. Then $Bw_i Bwb \subseteq Bw_i wB$, and $\{Bg \subseteq Bw_i Bwb\} = \{Bw_i wb' \subseteq Bw_i Bwb\}$. Suppose that $w^{-1}(r_i) < 0$. Then $w = w_i w'$ for some $w' \in W$ with $l(w) = l(w')+1$. Then

$Bw_i Bwb \subseteq BwB \cup Bw_i wB$, and $\{Bg \subseteq Bw_i Bwb\} = Bw_i wb \cup \{Bwb' \subseteq Bw_i Bw'b: Bwb' \neq Bwb\}$.

Now consider the coefficients of the f_{Bwb} on the left hand side of equation *. Suppose $w^{-1}(r_i) > 0$. Then the coefficient of f_{Bwb} is $q^{c_i}k_{Bwb} - \sum_{Bw_iwb' \subseteq Bw_iBwb} k_{Bw_iwb'}$.

Since $\{f_{Bvb}: v \in W,\ Bvb \subseteq BvB\}$ is a basis of $L(q)$, the coefficients of the f_{Bvb} occurring on the left hand side of * must all be zero. Thus, if $w^{-1}(r_i) > 0$,

$$q^{c_i}\, k_{Bwb} - \sum_{Bw_iwb' \subseteq Bw_iBwb} k_{Bw_iwb'} = 0. \qquad \text{(a)}$$

Now choose any coset $Bw_iwb' \subseteq Bw_iBwb$. Since $w^{-1}(r_i) > 0$, $(w_iw)^{-1}(r_i) < 0$, and so the coefficient of $f_{Bw_iwb'}$ on the left hand side of * is $q^{c_i}\, k_{Bw_iwb'} - k_{Bwb'} - \sum_{\substack{Bg \subseteq Bw_iBwb' \\ Bg \neq Bw_iwb'}} k_{Bg}$.

$$\text{So } (q^{c_i}+1)k_{Bw_iwb'} - k_{Bwb'} - \sum_{Bw_iwb'' \subseteq Bw_iBwb'} k_{Bw_iwb''} = 0. \qquad \text{(b)}$$

Since $Bw_iwb' \subseteq Bw_iBwb$, $Bwb' = Bwb$, and $\{Bw_iwb'' \subseteq Bw_iBwb'\}$ $= \{Bw_iwb'' \subseteq Bw_iBwb\}$. Then (b) can be written:

$$(q^{c_i}+1)k_{Bw_iwb'} - k_{Bwb} - \sum_{Bw_iwb'' \subseteq Bw_iBwb} k_{Bw_iwb''} = 0 \qquad \text{(c)}$$

Subtract (c) from (a):

$$(q^{c_i}+1)k_{Bwb} - (q^{c_i}+1)k_{Bw_iwb'} = 0.$$

Since $q^{c_i}+1 \neq 0$, $k_{Bwb} = k_{Bw_iwb'}$. Hence for all $w^{-1} \in X_{\{w_i\}}$, for any $Bwb \subseteq BwB$, $k_{Bwb} = k_{Bw_iwb'}$ for all $Bw_iwb' \subseteq Bw_iBwb$.

Now let $w^{-1} \in X_J$. Then for any $w' \in W_J$, we have $l(w'w) = l(w')+l(w)$. Let $y \in W$. Then $y = w_Jx$ for some $w_J \in W_J$ and some $x^{-1} \in X_J$ with $l(y) = l(w_J)+l(x)$. Let $w_J = w_{i_1}\cdots w_{i_s}$ be a reduced expression for w_J; for $1 \leq j \leq s$, $w_{i_j} \in W_J$. Choose any coset $Bw_Jwb' \subseteq Bw_JBwb$: then $Bwb' = Bwb$ and by the above

$$k_{Bw_Jwb'} = k_{Bw_{i_2}..w_{i_s}wb'} = k_{Bw_{i_3}..w_{i_s}wb'} = .. = k_{Bwb'} = k_{Bwb}.$$

Now $E_Jf_{Bwb} = \frac{1}{f(q)_J} \sum_{w_J \in W_J} \sum_{Bg \subseteq Bw_JBwb} f_{Bg}$, and so

$f = \sum_{w^{-1} \in X_J} \sum_{Bwb \subseteq BwB} k_{Bwb}E_Jf(q)_Jf_{Bwb}$. Conversely for all

$w^{-1} \in X_J$ and all $Bwb \subseteq BwB$, $E_Jf_{Bwb} \in E_JL(q)$. Now for any

$w^{-1} \in X_J$, f_{Bwb} occurs with non-zero coefficient in E_Jf_{Bwb}

but in no other $E_Jf_{Bw'b'}$, where $(w')^{-1} \in X_J$, with $Bw'b' \neq Bwb$.

(See (1.3.2) for this.) Hence $\{E_Jf_{Bwb}:w^{-1} \in X_J,\ Bwb \subseteq BwB\}$

is a set of linearly independent elements which generate

$E_JL(q)$, and so is a basis of $E_JL(q)$. Thus

$\dim E_JL(q) = \sum_{w \in X_J} q^{c_w}$ as $c_w = c_{w^{-1}}$ for all $w \in W$.

Finally, let ${}^G/G_J$ denote the set of right cosets of

G_J in G and let $L_J(q)$ be the set of functions $f: {}^G/G_J \to K$.

$L_J(q)$ becomes a left KG-module by defining for all $x \in G$

the function $xf \in L_J(q)$ given by

$$(xf)(G_Jg) = f(G_Jgx) \text{ for all } G_Jg \in {}^G/G_J.$$

Let M_J be the principal KG_J-module. Then as in (7.1.1),

$L_J(q)$ is isomorphic as KG-module to $M_J{}^G$.

Now $L_J(q)$ has K-basis $\{f_{G_Jwb}:\ w^{-1} \in X_J,\ G_Jwb \subseteq G_JwB\}$

where f_{G_Jwb} is given by

$$f_{G_Jwb}(G_Jg) = \begin{cases} 1 \text{ if } G_Jg = G_Jwb \\ 0 \text{ if } G_Jg \neq G_Jwb \end{cases} \quad \text{for all } G_Jg \in {}^G/G_J.$$

Then the map $\phi: E_JL(q) \to L_J(q)$ given by $\phi(E_Jf_{Bwb}) = f_{G_Jwb}$

for all $w^{-1} \in X_J$, $Bwb \subsetneq BwB$, extended by linearity to $E_JL(q)$, is clearly an isomorphism of KG-modules.

(2) Similarly, by noting that for all $w \in W$, $w \neq 1$, $q^{c_w} = 0$ in K.

(7.1.8) PROPOSITION: (1) $O_JL(q) = \{f \in L(q): E_{\{w_i\}}f = 0$ for all $w_i \in J\} = \{f \in L(q): O_{\{w_i\}}f = f$ for all $w_i \in J\}$. $O_JL(q)$ has dimension $\sum_{w \in Z_J} q^{c_w}$, where $Z_J = \{w \in W: w(\prod_J) \subseteq \Phi^-\}$.

(2) $o_JL(0) = \{f \in L(0): e_{\{w_i\}}f = 0$ for all $w_i \in J\} = \{f \in L(0): o_{\{w_i\}}f = f$ for all $w_i \in J\}$. $o_JL(0)$ has dimension $\sum_{w \in Z_J} q^{c_w}$, where Z_J is given in (1).

Proof: (1) Clearly $O_JL(q) \leqslant \{f \in L(q): E_{\{w_i\}}f = 0$ for all $w_i \in J\}$. Conversely, let $f \in L(q)$ satisfy $E_{\{w_i\}}f = 0$ for all $w_i \in J$. Then $(1 + a_{w_i})f = 0$ for all $w_i \in J$; that is, $a_{w_i}f = -f$ for all $w_i \in J$. Hence if $w \in W_J$, $a_wf = (-1)^{l(w)}f$, so $O_Jf = f$, and $f \in O_JL(q)$. Then

$$O_JL(q) = \{f \in L(q): E_{\{w_i\}}f = 0 \text{ for all } w_i \in J\}.$$

Similarly, if $O_{\{w_i\}}f = f$, then $(q^{c_i}1 - a_{w_i})f = (1+q^{c_i})f$, and so $a_{w_i}f = -f$, and so also

$$O_JL(q) = \{f \in L(q): O_{\{w_i\}}f = f \text{ for all } w_i \in J\}.$$

Suppose $f = \sum_{w \in W} \sum_{Bwb \subsetneq BwB} k_{Bwb}f_{Bwb} \in O_JL(q)$, where the $k_{Bwb} \in K$. Then $E_{\{w_i\}}f = 0$ for all $w_i \in J$ and so

$$f + \sum_{w \in W} \sum_{Bwb \subsetneq BwB} k_{Bwb}a_{w_i}f_{Bwb} = 0 \text{ for all } w_i \in J. \text{ That is,}$$

$$f + \sum_{w \in W} \sum_{Bwb \subsetneq BwB} \sum_{Bg \subsetneq Bw_iBwb} k_{Bwb}f_{Bg} = 0 \text{ for all } w_i \in J.$$

Now if $w^{-1}(r_i) > 0$, then $Bw_iBwb \subseteq Bw_iwB$, and $(w_iw)^{-1}(r_i) < 0$. If $w^{-1}(r_i) < 0$, then $w = w_iw'$ for some $w' \in W$ with $l(w) = l(w')+1$, and $(w')^{-1}(r_i) > 0$. Moreover, $\{Bw_iwb' \subseteq Bw_iBwb\} = Bw'b \cup \{Bwb' \subseteq Bw_iBw'b,\ Bwb' \neq Bwb\}$.

Suppose $w^{-1}(r_i) > 0$. Then f_{Bwb} occurs on the right side of the last equation with coefficient

$k_{Bwb} + \sum_{Bw_iwb' \subseteq Bw_iBwb} k_{Bw_iwb'}$. Since $\{f_{Bwb} : w \in W,\ Bwb \subseteq BwB\}$ are a basis of $L(q)$, this coefficient is zero. Thus $k_{Bwb} + \sum_{Bw_iwb' \subseteq Bw_iBwb} k_{Bw_iwb'} = 0$ for all $w \in W$, $w^{-1}(r_i) > 0$, $Bwb \subseteq BwB$, $w_i \in J$.

Suppose $w^{-1}(r_i) < 0$. Then as the coefficient of f_{Bwb} is zero, we have

$$k_{Bwb} + k_{Bw_iwb} + \sum_{\substack{Bwb' \subseteq Bw_iBw_iwb \\ Bwb' \neq Bwb}} k_{Bwb'} = 0.$$

i.e. $k_{Bw_iwb} + \sum_{Bwb' \subseteq Bw_iBw_iwb} k_{Bwb'} = 0$.

But this is the same type of equation as above, as $(w_iw)^{-1}(r_i) > 0$.

Hence for all $w_i \in J$, for all $w^{-1} \in X_{\{w_i\}}$, for all $Bwb \subseteq BwB$, $k_{Bwb} + \sum_{Bw_iwb' \subseteq Bw_iBwb} k_{Bw_iwb'} = 0$.

Let $x^{-1} \in X_J$ and $w \in W_J$, $w \neq w_{oJ}$. There exists $w_j \in J$ such that $(wx)^{-1}(r_j) > 0$, and so for any $Bwxb \subseteq BwxB$,

$$k_{Bwxb} + \sum_{Bw_jwxb' \subseteq Bw_jBwxb} k_{Bw_jwxb'} = 0.$$

Do the same for each $Bw_jwxb' \subseteq Bw_jBwxb$, provided $w_jw \neq w_{oJ}$.

This process terminates, and if $w_{oJ} = w'w$, where $l(w_{oJ}) = l(w')+l(w)$, we have

$$k_{Bwxb} + (-1)^{l(w')+1} \sum_{Bw_{oJ}xb' \subseteq Bw'Bwxb} k_{Bw_{oJ}xb'} = 0.$$

Thus if $w \in W$, $w = w_J x$ with $x^{-1} \in X_J$, $w_J \in W_J$, and $l(w) = l(w_J)+l(x)$, then for any $Bwb \subseteq BwB$, k_{Bwb} can be expressed as a linear combination of $\{k_{Bw_{oJ}xb'}: Bw_{oJ}xb' \subseteq Bw_{oJ}w_J Bw_J xb\}$. Now for each $x^{-1} \in X_J$, for each $Bw_{oJ}xb \subseteq Bw_{oJ}xB$, define $F_{Bw_{oJ}xb} = \sum_{w \in W_J} (-1)^{l(w)} f_{Bwxb}$.

Then
$$f = \sum_{x^{-1} \in X_J} \sum_{Bw_{oJ}xb \subseteq Bw_{oJ}xB} (-1)^{l(w_{oJ})} k_{Bw_{oJ}xb} F_{Bw_{oJ}xb}.$$

Conversely, for all $w_i \in J$, $E_{\{w_i\}} F_{Bw_{oJ}xb} = 0$, for all $x^{-1} \in X_J$ and all $Bw_{oJ}xb \subseteq Bw_{oJ}xB$. Hence $\{F_{Bw_{oJ}xb}: x^{-1} \in X_J, Bw_{oJ}xb \subseteq Bw_{oJ}xB\}$ is a basis of $O_J L(q)$, and so $\dim O_J L(q) = \sum_{x^{-1} \in X_J} q^{c_{w_{oJ}x}}$. Now clearly if $x^{-1} \in X_J$ then $(w_{oJ}x)^{-1}(\Pi_J) \subseteq \Phi^-$. Conversely, suppose that $w \in W$ and $w^{-1}(\Pi_J) \subseteq \Phi^-$. Then $(w_{oJ}w)^{-1} \in X_J$, and so $w = w_{oJ}x^{-1}$ for some $x \in X_J$. Thus $\dim O_J L(q) = \sum_{w \in Z_J} q^{c_w}$.

(2) Done similarly.

(7.1.9) PROPOSITION: (1) $E_J L(q) = \sum^{\oplus}_{L \supseteq J} E_L O_L^{\wedge} L(q)$ and $O_J L(q) = \sum^{\oplus}_{L \supseteq J} O_L E_L^{\wedge} L(q)$ for all $J \subseteq R$.

(2) $e_J L(0) = \sum^{\oplus}_{L \supseteq J} e_L o_L^{\wedge} L(0)$ and $o_J L(0) = \sum^{\oplus}_{L \supseteq J} o_L e_L^{\wedge} L(0)$ for all $J \subseteq R$.

Proof: (1) For each $L \supseteq J$, $E_L O_L^{\wedge} L(q) \leq E_J L(q)$. So $\sum_{L \supseteq J}^{\oplus} E_L O_L^{\wedge} L(q) \leq E_J L(q)$. But $E_J L(q) = E_J H_K(q) L(q)$ and so

$$E_J L(q) = \left(\sum_{L \supseteq J}^{\oplus} E_L O_L^{\wedge} H_K(q)\right) L(q) \leq \sum_{L \supseteq J}^{\oplus} E_L O_L^{\wedge} H_K(q) L(q)$$

$$= \sum_{L \supseteq J}^{\oplus} E_L O_L^{\wedge} L(q).$$

Hence $E_J L(q) = \sum_{L \supseteq J}^{\oplus} E_L O_L^{\wedge} L(q)$. Similarly, as $O_J H_K(q) = \sum_{L \supseteq J}^{\oplus} O_L E_L^{\wedge} H_K(q)$, we get $O_J L(q) = \sum_{L \supseteq J}^{\oplus} O_L E_L^{\wedge} L(q)$.

(2) Done similarly as $e_J H_K(0) = \sum_{L \supseteq J}^{\oplus} e_L o_L^{\wedge} H_K(0)$ and $o_J H_K(0) = \sum_{L \supseteq J}^{\oplus} o_L e_L^{\wedge} H_K(0)$.

(7.1.10) COROLLARY: (1) For all $J \subseteq R$,

$$\dim E_J O_J^{\wedge} L(q) = \sum_{w \in Y_J} q^{c_w} \text{ and } \dim e_J o_J^{\wedge} L(0) = \sum_{w \in Y_J} q^{c_w}.$$

(2) For all $J \subseteq R$,

$$\dim O_J^{\wedge} E_J L(q) = \sum_{w \in Y_J} q^{c_w} \text{ and } \dim o_J^{\wedge} e_J L(0) = \sum_{w \in Y_J} q^{c_w}.$$

Proof: (1) $\dim E_J L(q) = \sum_{w \in X_J} q^{c_w}$. We show by decreasing induction on $|J|$ that $\dim E_J O_J^{\wedge} L(q) = \sum_{w \in Y_J} q^{c_w}$.

Suppose $J = R$. Then $E_R L(q) = E_R O_\emptyset L(q)$ has dimension $\sum_{w \in X_R} q^{c_w}$. But $X_R = Y_R$ and so $\dim E_R O_\emptyset L(q) = \sum_{w \in Y_R} q^{c_w}$.

Suppose $|J| < |R|$. $E_J L(q) = \sum_{L \supseteq J}^{\oplus} E_L O_L^{\wedge} L(q)$. By induction, $\dim E_L O_L^{\wedge} L(q) = \sum_{w \in Y_L} q^{c_w}$ for all $L \supset J$. Hence

$$\dim E_J O_J^{\wedge} L(q) = \dim E_J L(q) - \sum_{L \supset J} \dim E_L O_L^{\wedge} L(q)$$

So $\dim E_J O_{\hat J} L(q) = \sum_{w \in X_J} q^{c_w} - \sum_{L \supset J} \sum_{y \in Y_L} q^{c_y}$

$= \sum_{w \in Y_J} q^{c_w}$, as $X_J = \bigcup_{L \supseteq J} Y_L$, a disjoint union. Similarly, $\dim e_J o_{\hat J} L(0) = \sum_{w \in Y_J} q^{c_w}$.

(2) As $\dim O_J L(q) = \sum_{w \in Z_J} q^{c_w}$, and

$O_J L(q) = \sum^{\oplus}_{L \supseteq J} O_L E_{\hat L} L(q)$, we show by decreasing induction on $|J|$ that $\dim O_J E_{\hat J} L(q) = \sum_{w \in Y_{\hat J}} q^{c_w}$.

If $J = R$, $Z_R = Y_\emptyset$ and $O_R L(q) = O_R E_\emptyset L(q)$ and so $\dim O_R L(q) = \sum_{w \in Y_\emptyset} q^{c_w}$. Suppose that $J \subset R$. Then

$Z_J = \bigcup_{L \subseteq \hat J} Y_L = \bigcup_{\hat L \supseteq J} Y_L$, both being disjoint unions.

So $\dim O_J E_{\hat J} L(q) = \dim O_J L(q) - \sum_{L \supset J} \dim O_L E_{\hat L} L(q)$

$= \sum_{w \in Y_{\hat J}} q^{c_w}$, by the above comments on Z_J.

(7.1.11) <u>COROLLARY</u>: (1) $E_J O_{\hat J} L(q) \cong E_J L(q) / \sum_{L \supset J} E_L L(q)$ and $e_J o_{\hat J} L(0) \cong e_J L(0) / \sum_{L \supset J} e_L L(0)$ for all $J \subseteq R$.

(2) $O_J E_{\hat J} L(q) \cong O_J L(q) / \sum_{L \supset J} O_L L(q)$ and $o_J e_{\hat J} L(0) \cong o_J L(0) / \sum_{L \supset J} o_L L(0)$ for all $J \subseteq R$.

(7.1.12) <u>COROLLARY</u>: (1) $E_J O_{\hat J} L(q)$ affords the character

$$\mu_J = \sum_{L \supseteq J} (-1)^{|L-J|} (1_{G_L})^G$$

of $G = G(q)$ over K.

(2) $e_J o_{\hat{J}} L(0)$ affords the

character $\mu_J = \sum_{L \supseteq J} (-1)^{|L-J|} (1_{G_L})^G$ of $G = G(q)$ over K.

Proof: (1) Use decreasing induction on $|J|$.

Suppose $J = R$. Then $E_R L(q) = \{f \in L(q) : f(Bg) = f(Bg')$ for all $Bg, Bg' \in {}^G/B\} = \{$constant functions of $L(q)\}$.

So $E_R L(q)$ affords the character 1_G of G; i.e. $u_R = 1_G = (1_G)^G$.

Suppose $|J| < |R|$. Then

$$E_J L(q) = E_J O_{\hat{J}} L(q) \oplus \sum_{L \supset J}^{\oplus} E_L O_{\hat{L}} L(q).$$

Now $E_J L(q)$ affords the character $(1_{G_J})^G$ of G by (7.1.7), and by induction $E_L O_{\hat{L}} L(q)$ affords the character μ_L of G. Hence $E_J O_{\hat{J}} L(q)$ affords the character μ_J' of G, where

$$\begin{aligned} \mu_J' &= (1_{G_J})^G - \sum_{L \supset J} \mu_L \\ &= (1_{G_J})^G - \sum_{L \supset J} \sum_{M \supseteq L} (-1)^{|M-L|} (1_{G_M})^G. \end{aligned}$$

Now if $M \supset J$, the coefficient of $(1_{G_M})^G$ in this is

$$- \sum_{\substack{L \\ J \subset L \subseteq M}} (-1)^{|M-L|}.$$

Suppose $M = J \cup \{w_{i_1}, \dots w_{i_r}\}$, with $w_{i_j} \in \hat{J}$ for all j, $1 \leqslant j \leqslant r$.

$$\begin{aligned} \text{Then } - \sum_{\substack{L \\ J \subset L \subseteq M}} (-1)^{|M-L|} &= 1 - r + \frac{r(r-1)}{2} - \dots + (-1)^{r-1}\, {}^rC_{r-1} \\ &\quad + (-1)^r\, {}^rC_r - (-1)^r\, {}^rC_r \\ &= (1-1)^r + (-1)^{r-1} = (-1)^{r-1} \end{aligned}$$

where $r = |M-J|$ and for any positive integer $k \leqslant r$,

${}^rC_k = \dfrac{(r)!}{(r-k)!\,(k)!}$. Then $\mu_J' = u_J = \sum_{L \supseteq J} (-1)^{|L-J|} (1_{G_L})^G$.

(2) Done similarly.

(7.1.13) COROLLARY: If μ_J is the character of G given in (1) or (2) of (7.1.12), then for all $J \subseteq R$,

$$(1_{G_J})^G = \sum_{L \supseteq J} \mu_L.$$

In particular, if $J = \emptyset$, then $(1_B)^G = \sum_{J \subseteq R} \mu_J$.

Proof: Use the inversion formula, (1.4.4).

(7.1.14) THEOREM: (1) $E_J O_{\hat{J}} L(q) \cong O_{\hat{J}} E_J L(q)$ as left KG-modules, for all $J \subseteq R$.

(2) $e_J o_{\hat{J}} L(0) \cong o_{\hat{J}} e_J L(0)$ as left KG-modules, for all $J \subseteq R$.

Proof: (1) From (5.19), the map $\Psi_J : E_J O_{\hat{J}} H_K(q) \to O_{\hat{J}} E_J H_K(q)$ given by left multiplication by $O_{\hat{J}}$ is an isomorphism of right $H_K(q)$-modules, and $O_{\hat{J}} E_J O_{\hat{J}} H_K(q) = O_{\hat{J}} E_J H_K(q)$.

Define Ψ_J: $E_J O_{\hat{J}} L(q) \to O_{\hat{J}} E_J L(q)$ by left multiplication by $O_{\hat{J}}$. Ψ_J is well-defined and is a homomorphism of left KG-modules. Ψ_J is onto, as $O_{\hat{J}} E_J O_{\hat{J}} L(q) = O_{\hat{J}} E_J O_{\hat{J}} H_K(q) L(q) = O_{\hat{J}} E_J H_K(q) L(q) = O_{\hat{J}} E_J L(q)$. As $\dim E_J O_{\hat{J}} L(q) = \dim O_{\hat{J}} E_J L(q)$, Ψ_J is one-one. Hence for all $J \subseteq R$, $E_J O_{\hat{J}} L(q) \cong O_{\hat{J}} E_J L(q)$.

(2) Done similarly, using (4.4.13).

(7.1.15) PROPOSITION: If $J = R$ or $R - \{w_i\}$ for any $w_i \in R$,

(1) $E_J O_{\hat{J}} L(q) = \{f \in E_J L(q) : E_M f = 0 \text{ for all } M \supset J\}$, and

(2) $e_J o_{\hat{J}} L(0) = \{f \in e_J L(0) : e_M f = 0 \text{ for all } M \supset J\}$.

Proof: Clearly, for any $J \subseteq R$, we have that

$E_J O_J^{\wedge} L(q) \leqslant \{f \in E_J L(q) : E_M f = 0 \text{ for all } M \supset J\}$, and

$e_J o_J^{\wedge} L(0) \leqslant \{f \in e_J L(0) : e_M f = 0 \text{ for all } M \supset J\}$.

The result is obviously true if $J = R$. So suppose $J = R-\{w_i\}$ for some $w_i \in R$. Let $f \in E_J L(q) = E_R L(q) \oplus E_J O_J^{\wedge} L(q)$ by (7.1.9(1)). Suppose that $f = f_R + f_J$, where $f_R \in E_R L(q)$ and $f_J \in E_J O_J^{\wedge} L(q)$; assume $E_R f = 0$. Then clearly $E_R f_J = 0$, and so $E_R f_R = f_R = 0$. Hence $f = f_J$ and $E_J O_J^{\wedge} L(q) = \{f \in E_J L(q) : E_R f = 0\}$. Similarly, if $J = R-\{w_i\}$, $e_J o_J^{\wedge} L(0) = \{f \in e_J L(0) : e_R f = 0\}$.

For the rest of this section, we will consider both $L(q)$ and $L(0)$ together, writing them as L, and writing the corresponding idempotents E_J, O_J and e_J, o_J respectively as e_J, o_J, for all $J \subseteq R$. Then for all $J \subseteq R$, we have that $e_J o_J^{\wedge} L \leqslant \{f \in e_J L : e_M f = 0 \text{ for all } M \supset J\}$.

(7.1.16) <u>LEMMA</u>: If $f \in L$, $J \subseteq R$, then for all $g \in G_J$,

$$e_J f(Bg) = e_J f(B).$$

<u>Proof</u>: $e_J f \in e_J L$, and the result follows by (7.1.7), as we can consider $e_J f$ as a function on the right cosets of G_J in G.

Hence any element of $e_J L$ is determined by its values on the cosets Bwb, where $w^{-1} \in X_J$ and $Bwb \subseteq BwB$. Suppose $f \in e_J L$; then $e_J f = f$. Let $M \supset J$ and suppose that $e_M f = 0$. Since $e_M f \in e_M L$, $e_M f$ is determined by its values on the cosets Bwb, where $w^{-1} \in X_M$ and $Bwb \subseteq BwB$. Now let $w^{-1} \in X_M$

and $Bwb \subseteq BwB$. Then $e_M f(Bwb) = 0$ gives

$$\sum_{v \in W_M} \sum_{Bg \subseteq BvBwb} f(Bg) = 0.$$

Now each $v \in W_M$ has the form $v = v(1)v(2)$, where $v(1) \in W_J$, $(v(2))^{-1} \in X_J^M$, and $l(v) = l(v(1))+l(v(2))$. So the above equation becomes

$$\sum_{v^{-1} \in X_J^M} \sum_{w_J \in W_J} \sum_{Bg \subseteq Bw_J BvBwb} f(Bg) = 0; \text{ that is,}$$

$$\sum_{v^{-1} \in X_J^M} \sum_{Bg \subseteq BvBwb} \sum_{w_J \in W_J} \sum_{Bg' \subseteq Bw_J Bg} f(Bg') = 0$$

Since $e_J f = f$, $f(Bg) = f(Bg')$ if $gg'^{-1} \in G_J$, we have that

$$\sum_{w_J \in W_J} \sum_{Bg' \subseteq Bw_J Bg} f(Bg') = f(q)_J f(Bg), \text{ and so}$$

$$e_M f(Bwb) = f(q)_J \sum_{v^{-1} \in X_J^M} \sum_{Bg \subseteq BvBwb} f(Bg) = 0$$

Now if K is of characteristic 0, $f(q)_J \neq 0$ for all $J \subseteq R$, and if the characteristic of K is p, where $q = p^s$ for some positive integer s, then $f(q)_J = 1$ in K for all $J \subsetneq R$. Hence for all $w^{-1} \in X_M$ and all $Bwb \subseteq BwB$,

$$\sum_{v^{-1} \in X_J^M} \sum_{Bg \subseteq BvBwb} f(Bg) = 0 \qquad (*)$$

Now let $\mathcal{L}_J = \{f \in e_J L : e_M f = 0 \text{ for all } M \supset J\}$. Then if $f \in \mathcal{L}_J$, $w^{-1} \in X_M$ for some $M \supset J$ and $Bwb \subseteq BwB$, then

$$\sum_{v^{-1} \in X_J^M} \sum_{Bg \subseteq BvBwb} f(Bg) = 0.$$

Suppose $\hat{J} = \{w_1, \ldots w_r\}$, and let $M_i = J \cup \{w_i\}$, $1 \leq i \leq r$. Then for each M_i and for each coset $Bwb \subseteq BwB$ where $w^{-1} \in X_{M_i}$,

$$\sum_{v^{-1} \in X_J^{M_i}} \sum_{Bg \subseteq BvBwb} f(Bg) = 0 \qquad \boxed{A}$$

Since $\mathcal{L}_J \leqslant e_J L$, we may consider any $f \in \mathcal{L}_J$ as a function on the cosets of G_J in G, and then $\boxed{A}$ becomes:

$$\sum_{v^{-1} \in X_J^{M_i}} \sum_{G_J g \subseteq G_J vBwb} f(G_J g) = 0, \text{ for all } Bwb \subseteq BwB$$

$$\text{where } w^{-1} \in X_{M_i} \qquad \boxed{B}.$$

Consider the Tits complex Δ_J of G with respect to G_J, as in (6.1), and in particular, consider the homology module $H_{r-1}(\Delta_J) \otimes K$. Suppose $|\hat{J}| \geqslant 2$. By (6.1.17) $c = \sum_{w \in X_J} \sum_{bwB \subseteq BwB} k^c_{bw}\, bw\sigma_{\hat{J}}$, where the $k^c_{bw} \in K$, is an (r-1)-cycle of Δ_J (with coefficients in K), if and only if for each M_i, for each $w \in X_{M_i}$ and for each $bwB \subseteq BwB$,

$$\sum_{v \in X_J^{M_i}} \sum_{gG_J \subseteq bwBvG_J} k^c_g = 0 \qquad \boxed{C}$$

Comparing $\boxed{B}$ and $\boxed{C}$ for each $v \in X_J^{M_i}$ and each $w \in X_{M_i}$, we have that $f(G_J v^{-1} w^{-1} b^{-1})$ satisfies the same equations as k^c_{bwv}. Hence for each (r-1)-cycle $c = \sum_{w \in X_J} \sum_{bwB \subseteq BwB} k^c_{bw}\, bw\sigma_{\hat{J}}$ of Δ_J, define the function $f^c \in \mathcal{L}_J$ by $f^c(G_J v^{-1} w^{-1} b^{-1}) = k^c_{bwv}$ for all $w \in X_{M_i}$, $v \in X_J^{M_i}$, and $bwvB \subseteq BwvB$. Similarly, given any function $f \in \mathcal{L}_J$, we can define an (r-1)-cycle $c(f) \in \Delta_J$ by

$c(f) = \sum_{w \in X_J} \sum_{bwB \subseteq BwB} k^c_{bw}\, bw\sigma_{\hat{J}}$ where for all $w \in X_J$ and all $bwB \subsetneq BwB$, $k^c_{bw} = f(G_J w^{-1} b^{-1})$. Then if $c_1,\ldots c_t$ are a basis of $H_{r-1}(\Delta_J) \otimes K$, $f^{c_1},\ldots f^{c_t}$ must be linearly independent elements of $\mathcal{L}_J$, and if $f_1,\ldots f_s$ are a basis of $\mathcal{L}_J$, then $c(f_1),\ldots c(f_s)$ are linearly independent $(r-1)$-cycles. Thus $\dim \mathcal{L}_J = \dim H_{r-1}(\Delta_J) \otimes K = \dim e_J o_{\hat{J}} L$, where $|\hat{J}| \geq 2$. Hence if $|\hat{J}| \geq 2$,

$$e_J o_{\hat{J}} L = \{f \in e_J L : e_M f = 0 \text{ for all } M \supset J\}. \text{ So:}$$

(7.1.17) <u>THEOREM</u>: For all $J \subsetneq R$,

$$e_J o_{\hat{J}} L = \{f \in e_J L : e_M f = 0 \text{ for all } M \supset J\}$$

(7.1.18) <u>THEOREM</u>: For all $J \subsetneq R$, where $|\hat{J}| \geq 2$, $|\hat{J}| = r$, $e_J o_{\hat{J}} L$ and $H_{r-1}(\Delta_J) \otimes K$ are isomorphic KG-modules.

<u>Proof</u>: We have a K-space isomorphism

$$T_J : H_{r-1}(\Delta_J) \otimes K \rightarrow e_J o_{\hat{J}} L$$

given as follows: if $c = \sum_{w \in X_J} \sum_{bwB \subsetneq BwB} k^c_{bw}\, bw\sigma_{\hat{J}}$,

where the $k^c_{bw} \in K$, and $c \in H_{r-1}(\Delta_J) \otimes K$, define $T_J(c) = f^c$, where for all cosets $G_J w^{-1} b^{-1}$ of G_J in G, $f^c(G_J w^{-1} b^{-1}) = k^c_{bw}$, $w \in X_J$. Then

$$f^c = \sum_{w \in X_J} \sum_{bwB \subsetneq BwB} k^c_{bw}\, f_{G_J w^{-1} b^{-1}},$$ a linear combination

of the characteristic functions of $e_J L$.

Further, T_J is a KG-module isomorphism, for if $g \in G$, $c \in H_{r-1}(\Delta_J) \otimes K$, $c = \sum_{w \in X_J} \sum_{bwB \subsetneq BwB} k^c_{bw}\, bw\sigma_{\hat{J}}$, then

$$gc = \sum_{w \in X_J} \sum_{bwB \subsetneq BwB} k^c_{bw}\, gbw\sigma_{\hat{J}}\text{, that is,}$$

$$gc = \sum_{w \in X_J} \sum_{bwB \subseteq BwB} k^c_{g^{-1}bw}\, bw\sigma_{\hat{J}}.$$ Then

$$gf^c = \sum_{w \in X_J} \sum_{bwB \subseteq BwB} k^c_{bw}\, gf_{G_J w^{-1} b^{-1}}$$

$$= \sum_{w \in X_J} \sum_{bwB \subseteq BwB} k^c_{bw}\, f_{G_J w^{-1} b^{-1} g^{-1}}$$

$$= \sum_{w \in X_J} \sum_{bwB \subseteq BwB} k^c_{g^{-1}bw}\, f_{G_J w^{-1} b^{-1}} = f^{gc}.$$

(7.1.19) <u>COROLLARY</u>: If K is a field of characteristic p and G is a finite group with a split (B,N) pair of characteristic p, then for all $J \subseteq R$ with $|\hat{J}| = r \geqslant 2$, $H_{r-1}(\Delta_J) \otimes K$ is an indecomposable KG-module.

Let G_J be a parabolic subgroup of G. Then

$$(1_B)^{G_J} = \sum_{L \subseteq J} \psi_L^{(J)}, \text{ where } \psi_L^{(J)} = \sum_{\substack{M \\ L \subseteq M \subseteq J}} (-1)^{|M-L|} (1_{G_M})^{G_J},$$

where the $\psi_L^{(J)}$ are ordinary characters of G_J if the characteristic of K is 0, and Brauer characters if the characteristic of K divides q, where $G_J = G_J(q)$.

Let $w_i \in \hat{J}$, and let $L_i = J \cup \{w_i\}$. Then $(1_B)^{G_{L_i}} \cong ((1_B)^{G_J})^{G_{L_i}}$ as KG_{L_i}-modules, and

$$(1_B)^{G_{L_i}} = \sum_{M \subseteq L_i} \phi_M^{(L_i)}, \text{ where}$$

$$\phi_M^{(L_i)} = \sum_{\substack{N \\ M \subseteq N \subseteq L_i}} (-1)^{|N-M|} (1_{G_N})^{G_{L_i}}.$$

(7.1.20) PROPOSITION: With notation as above,

$$(\psi_M^{(J)})^{G_{L_i}} = \phi_M^{(L_i)} + \phi_{M \cup \{w_i\}}^{(L_i)}$$

Proof:
$$(\psi_M^{(J)})^{G_{L_i}} = \sum_{\substack{N \\ M \subseteq N \subseteq J}} (-1)^{|N-M|} ((1_{G_N})^{G_J})^{G_{L_i}}$$

$$= \sum_{\substack{N \\ M \subseteq N \subseteq J}} (-1)^{|N-M|} (1_{G_N})^{G_{L_i}}$$

$$= \sum_{\substack{N \\ M \subseteq N \subset L_i \\ w_i \not\subset N}} (-1)^{|N-M|} (1_{G_N})^{G_{L_i}}$$

$$= \sum_{\substack{N \\ M \subseteq N \subseteq L_i}} (-1)^{|N-M|} (1_{G_N})^{G_{L_i}}$$

$$+ \sum_{\substack{N \\ M \cup \{w_i\} \subseteq N \subseteq L_i}} (-1)^{|N-\{M \cup \{w_i\}\}|} (1_{G_N})^{G_{L_i}}$$

$$= \phi_M^{(L_i)} + \phi_{M \cup \{w_i\}}^{(L_i)}$$

(7.2) <u>On the Composition Factors of $L(O)$.</u>

We restrict attention in this section to the case where G is a finite group with a split (B,N) pair (G,B,N,R,U) of rank n and characteristic p, and with Weyl group W. Let K be a field of characteristic p. Then

$$L = L(O) = \sum_{J \subseteq R}^{\oplus} e_J o_{\hat{J}} L,$$

a direct sum of indecomposable KG-modules. This decomposition arises from the direct sum decomposition of $A = En_{KG}(L)$ as $A = \sum^{\oplus}_{J \subseteq R} e_J o_{\hat{J}} A$, where $e_J o_{\hat{J}} A$ is a right A-module of dimension $|Y_J|$ over K and basis $\{e_J o_{\hat{J}} a_{y^{-1}} : y \in Y_J\}$. List the elements $y_1, .. y_s$ of Y_J in order of increasing length; if $i < j$ then $l(y_i) \leq l(y_j)$. Let $A_{J;i} = \{\sum_{j=i}^{s} k_j e_J o_{\hat{J}} a_{y_j^{-1}} : k_j \in K\}$.

Then $A_{J,1} = e_J o_{\hat{J}} A > A_{J,2} > .. > A_{J,s} > 0$ is a composition series for $e_J o_{\hat{J}} A$ of right A-modules.

Consider the corresponding series of KG-submodules of L: $e_J o_{\hat{J}} L = A_{J,1}L > A_{J,2}L > .. > A_{J,s}L > 0$ - (7.2.1)

(7.2.2) <u>**LEMMA**</u>: $A_{J,i}L/A_{J,i+1}L$ is generated as KG-module by $e_J o_{\hat{J}} a_{y_i^{-1}} f_B + A_{J,i+1}L$ for all i, $1 \leq i \leq s$, where $A_{J,s+1}L = 0$.

<u>Proof</u>: Result follows as f_B generates L as KG-module.

(7.2.3) <u>**LEMMA**</u>: For all i, $1 \leq i \leq n$, $a_{w_i} f_B = S_i f_B$, where S_i is defined in (2.3.2).

<u>Proof</u>: We show both $a_{w_i} f_B$ and $S_i f_B$ are functions in L

which take value 1 on Bw_iB and 0 outside, and hence are equal.

$$a_{w_i}f_B(Bg) = \sum_{Bg' \subseteq Bw_iBg} f_B(Bg')$$

$$= \begin{cases} 1 & \text{if } B \subseteq Bw_iBg \\ 0 & \text{otherwise.} \end{cases}$$

Now $B \subseteq Bw_iBg$ if and only if $Bg \subseteq Bw_iB$. Suppose $Bg \subseteq Bw_iB$; then for some $b \in B$, $Bg = Bw_ib$, and

$$Bw_iBw_ib \cap B = (Bw_iBw_i \cap B)b = (Bw_iB \cap Bw_i)w_ib$$

$$= Bw_iw_ib = B.$$

Thus $a_{w_i}f_B(Bg) = 1$ if and only if $Bg \subseteq Bw_iB$.

$S_if_B = \sum_{u \in X_i} un_if_B$, and so $S_if_B(Bg) = \sum_{u \in X_i} un_if_B(Bg)$

$$= \sum_{u \in X_i} f_B(Bgun_i).$$

Now $f_B(Bgun_i) = 0$ unless $Bgun_i = B$, i.e. unless $Bg = Bn_i^{-1}u^{-1}$, and then $g \in Bn_i^{-1}B = Bw_iB$. Conversely, suppose $g \in Bn_i^{-1}B$, and let $g \in Bn_i^{-1}u^{-1}$ for some $u \in X_i$. Then

$$S_if_B(Bg) = \sum_{u' \in X_i} f_B(Bn_i^{-1}u^{-1}u'n_i) = \sum_{u \in X_i} f_B(Bn_i^{-1}un_i).$$

$f_B(Bn_i^{-1}un_i) = 0$ unless $n_i^{-1}un_i \in B$. But $n_i^{-1}un_i \in X_i^{n_i}$, and $X_i^{n_i} = U^{n_i} \cap V \subset V$. So $f_B(Bn_i^{-1}un_i) = 0$ unless $n_i^{-1}un_i \in B \cap V = 1$. So $S_if_B(Bg) = f_B(B) = 1$. Hence S_if_B takes value 1 on $Bn_i^{-1}B = Bw_iB$ and 0 on $Bn_w^{-1}B$ if $w \neq w_i$.

The result now follows.

(7.2.4) <u>THEOREM</u>: ${}^{A}{}_{J,i}L/A_{J,i+1}L$ is generated as KG-module by a weight element of weight $(1_B, \{i_1, \ldots i_n\})$, where

$i_j = 0$ if $y_i^{-1}(r_j) > 0$ and $i_j = -1$ if $y_i^{-1}(r_j) < 0$: that is, if ${}^{A}J,i/A_{J,i+1}$ is isomorphic to the A-module affording the representation λ_L, where $y_i^{-1} \in Y_L$, then ${}^{A}J,i^{L}/A_{J,i+1}L$ is generated as KG-module by a weight element of weight $(1_B, \{i_1,..i_n\})$, where $i_j = \begin{cases} 0 \text{ if } w_j \in L \\ -1 \text{ if } w_j \in \hat{L}. \end{cases}$

<u>Proof</u>: ${}^{A}J,i^{L}/A_{J,i+1}L$ is generated as KG-module by $e_J o_{\hat{J}} a_{y_i^{-1}} f_B + A_{J,i+1}L$. For all $b \in B$,

$$b(e_J o_{\hat{J}} a_{y_i^{-1}} f_B + A_{J,i+1}L) = b(e_J o_{\hat{J}} a_{y_i^{-1}} f_B) + A_{J,i+1}L$$
$$= e_J o_J a_{y_i^{-1}} f_B + A_{J,i+1}L.$$

Hence $e_J o_{\hat{J}} a_{y_i^{-1}} f_B + A_{J,i+1}L$ is U-invariant and affords the character 1_B of B.

Now for each $w_j \in R$, consider

$$S_j(e_J o_{\hat{J}} a_{y_i^{-1}} f_B + A_{J,i+1}L) = S_j e_J o_{\hat{J}} a_{y_i^{-1}} f_B + A_{J,i+1}L$$
$$= e_J o_{\hat{J}} a_{y_i^{-1}} S_j f_B + A_{J,i+1}L \text{ as } S_j \in KG$$
$$= e_J o_{\hat{J}} a_{y_i^{-1}} a_{w_j} f_B + A_{J,i+1}L$$
$$= \begin{cases} -(e_J o_J a_{y_i^{-1}} f_B + A_{J,i+1}L) \text{ if } y_i^{-1}(r_j) < 0 \\ 0 \text{ if } y_i^{-1}(r_j) > 0. \end{cases}$$

From now on, we assume that K is a splitting field for the subgroup H of G.

(7.2.5) <u>COROLLARY</u>: ${}^{A}J,i^{L}/A_{J,i+1}L$ has a unique maximal submodule $A'_{J,i}L/A_{J,i+1}L$ such that

$$(A_{J,i}L/A_{J,i+1}L) \;/\; (A'_{J,i+1}L/A_{J,i+1}L) \cong M_{(1_B;i_1,..i_n)},$$

the irreducible KG-module of weight $(1_B;i_1,..i_n)$.

Proof: Use (2.4.21).

(7.2.6) THEOREM: Let G be a finite group with a split (B,N) pair (G,B,N,R,U) of rank n and characteristic p, and let W be the Weyl group of G. Let K be a field of characteristic p which is a splitting field for the subgroup H of G. Let M be the principal KB-module, and M^G the KG-module induced from M. Then the irreducible KG-module $M_{(1_B;J)} = M_{(1_B;\mu_1,..\mu_n)}$, where $\mu_i=0$ if $w_i \in J$ and $\mu_i=-1$ if $w_i \in \hat{J}$, occurs as a composition factor of M^G with multiplicity at least $|Y_J|$.

Proof: Consider the composition series for A obtained by taking a composition series for each $e_J o_{\hat{J}} A$, for all $J \subseteq R$. This gives us a corresponding series of KG-submodules. Let $A = A_1 > A_2 > A_3 > .. > A_{|W|} > 0$ be the composition series for A. Then each A_i/A_{i+1} is an irreducible right A-module, generated by $e_J o_{\hat{J}} a_y + A_{i+1}$ for some $J \subseteq R$ and some $y^{-1} \in Y_J$. This affords the representation λ_N of A, where $y \in Y_N$. The corresponding factor in the series of KG-modules, $A_iL/A_{i+1}L$, (as $L \cong M^G$ as left KG-module), is generated as KG-module by a weight vector

$$e_J o_{\hat{J}} a_y f_B + A_{i+1}L$$

of weight $(1_B;\mu_1,..\mu_n)$, where $\mu_i = 0$ if $w_i \in N$, and $\mu_i = -1$ if $w_i \in \hat{N}$, by (7.2.4). By (7.2.5), $A_iL/A_{i+1}L$ has as a composition factor the irreducible KG-module $M_{(1_B;\mu_1,..\mu_n)}$. Each such element $y \in Y_N$ gives rise to $M_{(1_B;\mu_1,..\mu_n)}$ in this way, and so $M_{(1_B;\mu_1,..\mu_n)} = M_{(1_B,N)}$ occurs in the composition series of M^G with multiplicity at least $|Y_N|$.

(7.2.7) <u>THEOREM</u>: With G and K as in (7.2.6), $e_J o_{\hat{J}} a_{(w_o w_{oJ})^{-1}} L$ is an irreducible submodule of L, where $w_o w_{oJ}$ is the unique element of maximal length in Y_J.

<u>Proof</u>: The KG-module $\mathcal{F}_{1_B}$ of (2.4.3) is the same as the KG-module L which we have discussed in this section, and the KG-endomorphism T_n of $\mathcal{F}_{1_B}$ is the same as the KG-endomorphism a_w of L, where $\theta(n) = w$. Hence for each $J \subseteq R$, the submodule $\Theta^J_{w_o} L$ is an irreducible KG-submodule of L, where $\Theta^J_{w_o}$ is defined as in (2.4.12) in terms of the elements $a_{w_i} \in A = En_{KG}(L)$ in place of the elements T_i for all $w_i \in R$.

We will prove the theorem by showing that for all $J \subseteq R$, $\Theta^{\bar{J}}_{w_o} L = e_J o_{\hat{J}} a_{(w_o w_{oJ})^{-1}} L$, where $\bar{J}$ is defined in (2.4.13).

Write $N = \bar{J}$. Then $\Theta^N_{w_o} L$ is an irreducible submodule of L, and by (2.4.14) and (2.4.17) we have

$$\Theta^N_{w_o} L \leqslant \{f \in L : a_{w_i} f = 0 \text{ for all } w_i \in J,$$
$$(1 + a_{w_i})f = 0 \text{ for all } w_i \in \hat{J}\}.$$

So $\Theta^N_{w_o} L \leq \{f \in e_J L : e_M f = 0 \text{ for all } M \supset J\} = e_J o_J^{\wedge} L$ by (7.1.17). Now $\Theta^N_{w_o} \in A$, and if $a \in A$ satisfies $af_B = 0$, then we must have $a = 0$. So $\Theta^N_{w_o} \in e_J o_J^{\wedge} A$. Let

$$\Theta^N_{w_o} = \sum_{y \in Y_J} k_y e_J o_J^{\wedge} a_{y^{-1}}, \quad k_y \in K.$$

By (2.4.17), $S_i \Theta^N_{w_o} f_B = \begin{cases} -\Theta^N_{w_o} f_B \text{ if } w_i \in \hat{N} \\ 0 \text{ if } w_i \in N. \end{cases}$

Thus since $S_i \Theta^N_{w_o} f_B = \Theta^N_{w_o} S_i f_B = \Theta^N_{w_o} a_{w_i} f_B$ for all $w_i \in R$,

we have $$\Theta^N_{w_o} a_{w_i} = \begin{cases} -\Theta^N_{w_o} \text{ if } w_i \in \hat{N} \\ 0 \text{ if } w_i \in N. \end{cases} \qquad (1)$$

Now, for all $y \in Y_J$,

$$e_J o_J^{\wedge} a_{y^{-1}} a_{w_i} = \begin{cases} -e_J o_J^{\wedge} a_{y^{-1}} \text{ if } y^{-1}(r_i) < 0 \\ 0 \text{ if } y^{-1}(r_i) = r_j \text{ for some } r_j \in \prod_J \\ e_J o_J^{\wedge} a_{(w_i y)^{-1}} \text{ if } y^{-1}(r_i) > 0 \text{ and } y^{-1}(r_i) \neq r_j \\ \qquad \text{for any } r_j \in \prod_J, \text{ with } w_i y \in Y_J. \end{cases}$$

So $$\Theta^N_{w_o} a_{w_i} = -\sum_{\substack{y \in Y_J \\ y^{-1}(r_i) < 0}} k_y e_J o_J^{\wedge} a_{y^{-1}} + \sum_{\substack{y \in Y_J \\ y^{-1}(r_i) > 0 \\ w_i y \in Y_J}} k_y e_J o_J^{\wedge} a_{(w_i y)^{-1}}. \qquad (2)$$

If $w_i \in \hat{N}$, then we must have, comparing (1) and (2), that $k_y = 0$ for all $y \in Y_J$ with $y^{-1}(r_i) > 0$. Hence

$$\Theta^N_{w_o} = \sum_{\substack{y \in Y_J \\ y^{-1}(\prod_{\hat{N}}) < 0}} k_y e_J o_J^{\wedge} a_{y^{-1}}.$$

Further, if $w_i \in N$, then comparing (1) and (2) we get

(a) $k_y = 0$ if $y^{-1}(r_i) < 0$, and $y^{-1}(r_i) = -r_j$ for some $r_j \in \prod_{\hat{J}}$.

(b) If $y^{-1}(r_i) < 0$ and $y^{-1}(r_i) \neq -r_j$ for any $r_j \in \prod_{\hat{J}}$, then $k_y = k_{w_i y}$.

(c) If $y^{-1}(r_i) > 0$ and $y^{-1}(r_i) \neq r_j$ for any $r_j \in \prod_J$, then $k_y = k_{w_i y}$.

Choose $y \in Y_J$ such that $k_y \neq 0$. Then $y^{-1}(\prod_{\hat{N}}) < 0$, and $y^{-1}(r_i) \neq -r_j$ for any $r_i \in \prod_N$, with $r_j \in \prod$. Hence for all $r_i \in \prod_N$, either $y^{-1}(r_i) < 0$ but $y^{-1}(r_i) \neq -r_j$ for any $r_j \in \prod_J$ or $y^{-1}(r_i) > 0$.

If $y^{-1}(r_i) < 0$ and $y^{-1}(r_i) \neq -r_j$, some $r_j \in \prod_{\hat{J}}$, for all $r_i \in \prod_N$, then $y^{-1}(\prod) < 0$ and so $y^{-1} = w_o$. So $y = w_o$ and $J = \emptyset$. In this case $\Theta_{w_o}^{\emptyset} = a_{w_o}$ and the result is true.

Hence suppose there exists $r_i \in \prod_N$ such that $y^{-1}(r_i) > 0$. If $y^{-1}(r_i) \neq r_j$ for any $r_j \in \prod_J$, then $k_y = k_{w_i y}$. Continuing in this way, we end up with an element $y_o \in Y_J$, $k_{y_o} \neq 0$, such that $y_o^{-1}(\prod_L) < 0$ for some $L \supseteq \hat{N}$, and if $r_i \in \prod_{\hat{L}}$, then $y_o^{-1}(r_i) = r_j$ for some $r_j \in \prod_J$. Then by (1.3.7(1)), y_o^{-1} is the unique element of maximal length in $Y_{\hat{L}}$, so $y_o^{-1} = w_o w_{o\hat{L}}$. Then $y_o = w_{o\hat{L}} w_o = w_o w_{o\hat{\hat{L}}}$. But $y_o \in Y_J$, and so $J = \hat{\hat{L}}$.

That is, $N = \bar{J} = \hat{L}$. So $y_o^{-1}(\prod_{\hat{N}}) < 0$ and if $r_i \in \prod_N$ then $y_o^{-1}(r_i) = r_j$ for some $r_j \in \prod_J$. In particular, there is no $r_i \in \prod_N$ with $y_o^{-1}(r_i) < 0$. Hence k_{y_o} is the only non-zero coefficient and

$$\Theta_{w_o}^{\bar{J}} = \Theta_{w_o}^{N} = e_J o_{\hat{J}} a_{(w_o w_{oJ})^{-1}} k, \text{ for some } k \in K.$$

Hence $e_J o_J^{\wedge} a_{(w_o w_{oJ})^{-1}} L$ is irreducible.

(7.2.8) <u>COROLLARY</u>: For all $J \subseteq R$, $\Theta_{w_o}^{J}$ is a scalar multiple of $e_{\bar{J}} o_{\bar{J}}^{\wedge} a_{(w_o w_{o\bar{J}})^{-1}}$.

We now know what $|W|$ of the composition factors of L are, but there may be others. We will give some examples later.

Consider the natural composition series of A, given by (4.2.4). Let

$$A = I_1 > I_2 > .. > I_{|W|} > 0$$

be this natural composition series. Then for all j, $1 \leqslant j \leqslant |W|$, I_j is a two-sided ideal of A and I_j/I_{j+1} is a left and a right A-module, where $I_{|W|+1} = 0$.

This gives us a series of KG-submodules of L which are also A-submodules of L:

$$L = I_1 L > I_2 L > .. > I_{|W|} L > 0.$$

If I_j/I_{j+1} is generated as A-module by $a_w + I_{j+1}$ for some $w \in W$, then $I_j L/I_{j+1} L$ is generated as A-module and as KG-module by $a_w f_B + I_{j+1} L$.

(7.2.9) <u>LEMMA</u>: $a_w f_B + I_{j+1} L$ is a weight vector of weight $(1_B;\mu_1,..\mu_n)$ where $\mu_i = 0$ if $w(r_i) > 0$ and $\mu_i = -1$ if $w(r_i) < 0$. Hence $I_j L/I_{j+1} L$ has a unique maximal KG-submodule $I_j' L/I_{j+1} L$ such that

$$(I_jL/I_{j+1}L) \;/\; (I_j'L/I_{j+1}L) \cong M_{(1_B;\mu_1,..\mu_n)}$$

as KG-modules.

Proof: $a_w f_B + I_{j+1}L$ is clearly U-invariant and affords the character 1_B of B.

$$\begin{aligned} S_i(a_w f_B + I_{j+1}L) &= S_i a_w f_B + I_{j+1}L \\ &= a_w S_i f_B + I_{j+1}L \\ &= a_w a_{w_i} f_B + I_{j+1}L \\ &= \begin{cases} -(a_w f_B + I_{j+1}L) & \text{if } w(r_i) < 0 \\ 0 & \text{if } w(r_i) > 0. \end{cases} \end{aligned}$$

Hence $a_w f_B + I_{j+1}L$ is a weight element of weight $(1_B;\mu_1,..\mu_n)$ where $\mu_i = 0$ if $w(r_i) > 0$ and $\mu_i = -1$ if $w(r_i) < 0$.

The rest follows by (2.4.21).

Hence using the natural composition series of A, we also get that $M_{(1_B;\mu_1,..\mu_n)}$ occurs as a composition factor of the KG-module M^G with multiplicity at least $|Y_J|$, where $J = \{w_i \in R: \mu_i = 0\}$.

EXAMPLES:

(1) Let $G = SL_2(p)$, the group of 2×2 matrices of determinant 1 over the field GF(p) of p elements; G has a split (B,N) pair with B(p) the subgroup of upper triangular matrices, N(p) the subgroup of monomial matrices, and H(p) the subgroup of diagonal matrices. $W = N(p)/H(p)$ and W is the Weyl group of type A_1; also $W \cong S_2$.

Let K be a field of characteristic p which is a splitting field for $H(p)$. Let L be the KG-module induced from the trivial $KB(p)$-module. Then

$$L = (1 + a_{w_1})L \oplus a_{w_1}L$$

where $(1 + a_{w_1})L$ is an irreducible KG-module of dimension 1 and weight $(1_B;0)$, and $a_{w_1}L$ is an irreducible KG-module of dimension p and weight $(1_B;-1)$.

(2) Let $G = SL_3(p)$, the group of 3×3 matrices of determinant one over the field $GF(p)$ of p elements. Then G has a split (B,N) pair, where B is the subgroup of upper triangular matrices, N is the subgroup of monomial matrices, H is the subgroup of diagonal matrices and the Weyl group W of G is of type A_2. Write $R = \{w_1,w_2\}$.

Let K be a field of characteristic p which is a splitting field for H, and let L be the KG-module induced from the principal KB-module. Then:

$$L = (1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1})L \oplus (1 + a_{w_1})a_{w_2}L$$
$$\oplus\ (1 + a_{w_2})a_{w_1}L \oplus a_{w_1w_2w_1}L.$$

Now, $(1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1})L$ is an irreducible KG-module of dimension 1 and weight $(1_B;0,0)$, and $a_{w_1w_2w_1}L$ is an irreducible KG-module of dimension p^3 and weight $(1_B;-1,-1)$.

$(1 + a_{w_1})a_{w_2}L$ is an indecomposable KG-module of dimension $p^2 + p$. It has an irreducible submodule $(1 + a_{w_1})a_{w_2}a_{w_1}L$ of weight $(1_B;-1,0)$. By unpublished work of Braden, $\dim M_{(1_B;-1,0)} = \frac{p^2 + p}{2}$ and $\dim M_{(1_B;0,-1)} = \frac{p^2 + p}{2}$. The factor module $(1 + a_{w_1})a_{w_2}L/(1 + a_{w_1})a_{w_2w_1}L$ contains an isomorphic copy of $M_{(1_B;0,-1)}$, and considering dimensions, $(1 + a_{w_1})a_{w_2}L/(1 + a_{w_1})a_{w_2w_1}L \cong M_{(1_B;0,-1)}$. So we have the following situation:

$$\begin{array}{rl}
 & \circ\ (1 + a_{w_1})a_{w_2}L \\
M_{(1_B;0,-1)} \left\{ \right. & | \\
 & \circ\ (1 + a_{w_1})a_{w_2w_1}L \\
M_{(1_B;-1,0)} \left\{ \right. & | \\
 & \circ\ 0
\end{array}$$

There is a similar situation for the indecomposable KG-module $(1 + a_{w_2})a_{w_1}L$, as follows:

$$\begin{array}{rl}
 & \circ\ (1 + a_{w_2})a_{w_1}L \\
M_{(1_B;-1,0)} \left\{ \right. & | \\
 & \circ\ (1 + a_{w_2})a_{w_1w_2}L \\
M_{(1_B;0,-1)} \left\{ \right. & | \\
 & \circ\ 0
\end{array}$$

In this example, there are only $|W|$ composition factors of L.

(3) Let $G = Sp_4(p)$, the group of 4×4 matrices T over the field GF(p) which satisfy $T^tET = E$, where $E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$.

G has a split (B,N) pair, and the Weyl group W of G is of type B_2.

Let K be a field of characteristic p which is a splitting field for the subgroup H of G, and let L be the KG-module induced from the principal KB-module. Then:

$$L = \left[1 + a_{w_1w_2w_1w_2}\right]L \oplus (1 + a_{w_1})a_{w_2}L \oplus (1 + a_{w_2})a_{w_1}L$$
$$\oplus\ a_{w_1w_2w_1w_2}L.$$

The irreducible KG-submodules of L are (for $p > 2$)

(a) $\left[1 + a_{w_1w_2w_1w_2}\right]L$, of dimension 1 and weight $(1_B;0,0)$,

(b) $(1 + a_{w_1})a_{w_2w_1w_2}L$, of weight $(1_B;0,-1)$ and dimension
$$\frac{p(p+1)(2p+1)}{6}$$

(c) $(1 + a_{w_2})a_{w_1w_2w_1}L$, of weight $(1_B;-1,0)$ and dimension
$$\frac{p(p+1)(p+2)}{6}$$

(d) $a_{w_1w_2w_1w_2}L$, of dimension p^4 and weight $(1_B;-1,-1)$.

So we need only consider the indecomposable submodules $(1 + a_{w_1})a_{w_2}L$ and $(1 + a_{w_2})a_{w_1}L$. We have a series of KG-submodules of $(1 + a_{w_1})a_{w_2}L$ as follows:

$$(1 + a_{w_1})a_{w_2}L > (1 + a_{w_1})a_{w_2w_1}L > (1 + a_{w_1})a_{w_2w_1w_2}L > 0.$$

The top factor, $(1 + a_{w_1})a_{w_2}L/(1 + a_{w_1})a_{w_2w_1}L$, contains a copy of $M_{(1_B;0,-1)}$, the middle factor $(1 + a_{w_1})a_{w_2w_1}L/(1 + a_{w_1})a_{w_2w_1w_2}L$ contains a copy of $M_{(1_B;-1,0)}$, and the bottom factor $(1 + a_{w_1})a_{w_2w_1w_2}L \cong M_{(1_B;0,-1)}$

But dim $(1 + a_{w_1})a_{w_2}L = p^3+p^2+p$, and

$$2 \dim M_{(1_B;0,-1)} + \dim M_{(1_B;-1,0)} = \frac{5p^3+9p^2+4p}{6} \text{ ; hence}$$

there are composition factors of dimension $\frac{p(p-1)(p-2)}{6}$ to be accounted for.

Similarly, $(1 + a_{w_2})a_{w_1}L$ contains $M_{(1_B;-1,0)}$ with multiplicity at least 2 and $M_{(1_B;0,-1)}$ with multiplicity at least 1. In this case there are composition factors of dimension $\frac{p(p-1)(2p-1)}{6}$ to be accounted for.

Now the irreducible KG-modules where $G = Sp_4(p)$ and $K = GF(p)$ are in one-one correspondence with points (a,b) in the restricted fundamental region $0 \leqslant a \leqslant p-1$, $0 \leqslant b \leqslant p-1$. Let $d_{(a,b)}$ be the dimension of the irreducible representation of the algebraic group $Sp_4(C)$ with dominant weight (a,b). $d_{(a,b)}$ is given by Weyl's dimension formula, and is as follows:

$$d_{(a,b)} = \frac{(a+1)(b+1)(a+b+2)(a+2b+3)}{6}$$

Let $M_{(a,b)}$ be the irreducible KG-module corresponding to the pair (a,b), $0 \leqslant a \leqslant p-1$, $0 \leqslant b \leqslant p-1$, and let $m(a,b) = \dim M_{(a,b)}$. If $p > 2$, then we have

$$m(0,0) = d_{(0,0)} = 1$$

$$m(p-1,p-1) = d_{(p-1,p-1)} = p^4$$

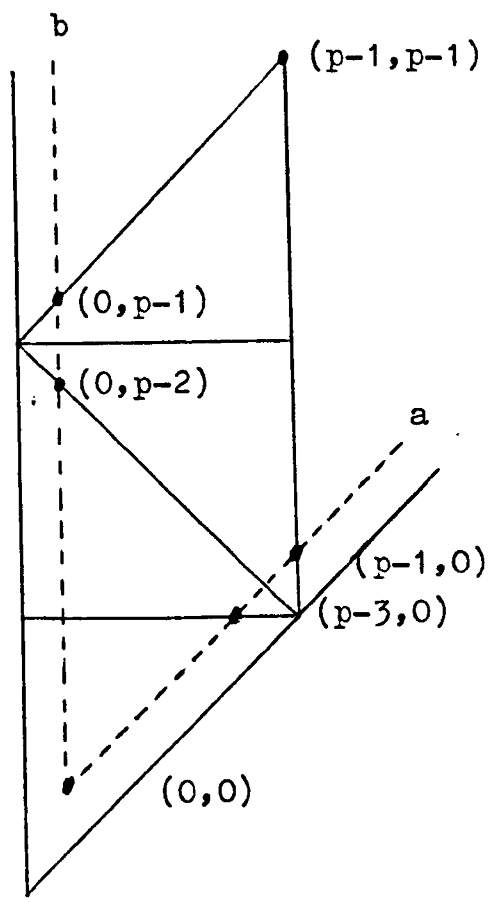
$$m(p-1,0) = d_{(p-1,0)} = \frac{p(p+1)(p+2)}{6}$$

$$m(0,p-1) = d_{(0,p-1)} = \frac{p(p+1)(2p+1)}{6}$$

$$m(p-3,0) = d_{(p-3,0)} = \frac{p(p-1)(p-2)}{6}$$

$$m(0,p-2) = d_{(0,p-2)} = \frac{p(p-1)(2p-1)}{6}$$

The corresponding points occur in the restricted fundamental region as follows:



From (2.4.22), $M_{(0,0)}$ corresponds to $M_{(1_B;0,0)}$,

$M_{(p-1,p-1)}$ corresponds to $M_{(1_B;-1,-1)}$,

$M_{(p-1,0)}$ corresponds to $M_{(1_B;-1,0)}$,

and $M_{(0,p-1)}$ corresponds to $M_{(1_B;0,-1)}$.

<u>Conjecture:</u> $(1 + a_{w_1})a_{w_2}L$ contains $M_{(p-3,0)}$ with multiplicity 1, and $(1 + a_{w_2})a_{w_1}L$ contains $M_{(0,p-2)}$ with multiplicity 1.

<u>Example</u>: When $p = 3$, $m(p-3,0) = m(0,0) = 1$, and

$m(0,p-2) = m(0,1) = 5$. Clearly $M_{(0,0)}$ is the only possibility for an extra factor of $(1 + a_{w_1})a_{w_2}L$.

(4) Let $G = SL_4(p)$, the group of 4×4 matrices of determinant one over the field $GF(p)$ of p elements. Then G has a split (B,N) pair, with B the subgroup of upper triangular matrices, N the subgroup of monomial matrices, H the subgroup of diagonal matrices and the Weyl group W of G is of type A_3. Write $R = \{w_1, w_2, w_3\}$.

Let K be a field of characteristic p which is a splitting field for H, and let L be the KG-module induced from the principal KB-module. Then:

$$\begin{aligned} L = {} & \left[1 + a_{w_1w_2w_3w_1w_2w_1}\right]L \oplus \left[1 + a_{w_1w_2w_1}\right]a_{w_3}L \\ & \oplus \left[1 + a_{w_1w_3}\right]a_{w_2}L \oplus \left[1 + a_{w_2w_3w_2}\right]a_{w_1}L \\ & \oplus (1 + a_{w_1})a_{w_2w_3w_2}L \oplus (1 + a_{w_2})a_{w_1w_3}L \\ & \oplus (1 + a_{w_3})a_{w_1w_2w_1}L \oplus a_{w_1w_2w_3w_1w_2w_1}L. \end{aligned}$$

Now $\left[1 + a_{w_1w_2w_3w_1w_2w_1}\right]L$ is irreducible, of dimension 1 and weight $(1_B;0,0,0)$, and $a_{w_1w_2w_3w_1w_2w_1}L$ is irreducible, of dimension p^6 and weight $(1_B;-1,-1,-1)$.

$\left[1 + a_{w_1w_2w_1}\right]a_{w_3}L$ has dimension p^3+p^2+p, and contains $M_{(1_B;0,0,-1)}$, $M_{(1_B;0,-1,0)}$ and $M_{(1_B;-1,0,0)}$.

$\left[1 + a_{w_1w_3}\right]a_{w_2}L$ has dimension $p^4+p^3+2p^2+p$, and contains $M_{(1_B;0,0,-1)}$, $M_{(1_B;0,-1,0)}$ twice, $M_{(1_B;-1,0,0)}$ and $M_{(1_B;-1,0,-1)}$.

$\left[1 + a_{w_2w_3w_2}\right]a_{w_1}L$ has dimension p^3+p^2+p, and contains $M_{(1_B;0,0,-1)}$, $M_{(1_B;0,-1,0)}$ and $M_{(1_B;-1,0,0)}$.

$(1 + a_{w_1})a_{w_2w_3w_2}L$ has dimension $p^5+p^4+p^3$, and contains $M_{(1_B;0,-1,-1)}$, $M_{(1_B;-1,0,-1)}$ and $M_{(1_B;-1,-1,0)}$.

$(1 + a_{w_2})a_{w_1w_3}L$ has dimension $p^5+2p^4+p^3+p^2$, and contains $M_{(1_B;0,-1,-1)}$, $M_{(1_B;-1,0,-1)}$ twice, $M_{(1_B;-1,-1,0)}$ and $M_{(1_B;0,-1,0)}$.

$(1 + a_{w_3})a_{w_1w_2w_1}L$ has dimension $p^5+p^4+p^3$, and contains $M_{(1_B;0,-1,-1)}$, $M_{(1_B;-1,0,-1)}$ and $M_{(1_B;-1,-1,0)}$.

We look at the cases when p=2 and p=3.

	$\underline{p=2}$	$\underline{p=3}$
dim $M_{(1_B;0,0,-1)}$	4	10
dim $M_{(1_B;0,-1,0)}$	6	19
dim $M_{(1_B;-1,0,0)}$	4	10
dim $M_{(1_B;0,-1,-1)}$	20	126
dim $M_{(1_B;-1,-1,0)}$	20	126
dim $M_{(1_B;-1,0,-1)}$	15	69

<u>The case p=2</u>

We must have that $(1 + a_{w_1})a_{w_2w_3w_2}L$, $(1 + a_{w_2})a_{w_1w_3}L$ and $(1 + a_{w_3})a_{w_1w_2w_1}L$ each contain two copies of

$M_{(1_B;0,0,0)}$. We then have all the composition factors in this case.

<u>The case p=3.</u>

$\left[1 + a_{w_1w_2w_1}\right]a_{w_3}L$ and $\left[1 + a_{w_2w_3w_2}\right]a_{w_1}L$ have no other factors other than those already given.

$\left[1 + a_{w_1w_3}\right]a_{w_2}L$ must also contain two copies of $M_{(1_B;0,0,0)}$. Both $(1 + a_{w_1})a_{w_2w_3w_2}L$ and $(1 + a_{w_3})a_{w_1w_2w_1}L$ have other factors of total dimension 30, and $(1 + a_{w_2})a_{w_1w_3}L$ has other factors of dimension totalling 32.

Appendix 1: CLASSIFICATION OF FINITE COXETER SYSTEMS.

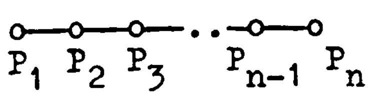
Let (W,R) be a finite Coxeter system. The graph D of (W,R) is a graph where the nodes are in one-one correspondence with the elements of R, and the number of bonds connecting nodes P_i and P_j is equal to $n_{ij}-2$, where n_{ij}= order of w_iw_j in W, for all w_i, $w_j \in R$. P_i and P_j are not connected if and only if $w_iw_j=w_jw_i$, $i\neq j$.

If (W,R) is indecomposable, then its graph $D=D(W)$ is connected. If (W,R) is not indecomposable, then its graph D is a disjoint union of connected components:

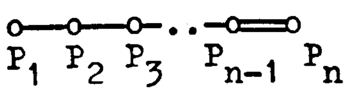
$$D = D_1 \cup D_2 \cup .. \cup D_s$$

where each D_i is a non-zero connected graph. Each D_i determines a subset J_i of R, and hence a parabolic subgroup W_{J_i} of W. Then, $W = W_{J_1} \times W_{J_2} \times .. \times W_{J_s}$, a direct product of subgroups.

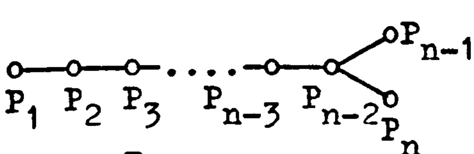
THEOREM: Let (W,R) be a finite indecomposable Coxeter system. Then the graph of (W,R) must be one of the following:



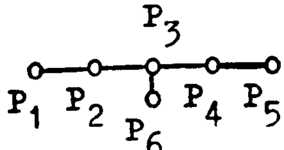
A_n, $n\geq 1$



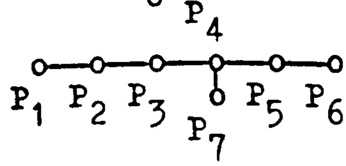
B_n, $n\geq 2$



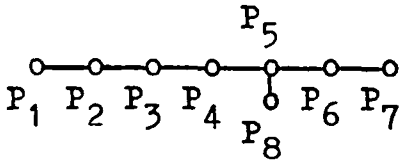
D_n, $n\geq 4$



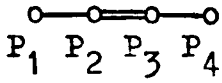
E_6



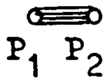
E_7



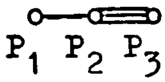
E_8



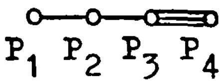
F_4



G_2



H_3



H_4

o--o (m-2) bonds

P_1 P_2

$I_2(m)$, $m=3$, or $m \geqslant 5$.

Appendix 2: SOME EXAMPLES OF FINITE COXETER GROUPS.

We give some examples of finite Coxeter groups, calculating for each the sets Y_J, the orders of the elements in Y_J, and the set $K \subset R$ such that $w^{-1} \in Y_K$, where $w \in Y_J$. Finally we list the conjugacy classes of the group.

The groups we discuss are:

(1) $W(A_1) \cong S_2$, $|W(A_1)| = 2$.

(2) $W(A_2) \cong S_3$, $|W(A_2)| = 6$.

(3) $W(A_3) \cong S_4$, $|W(A_3)| = 24$.

(4) $W(A_4) \cong S_5$, $|W(A_4)| = 120$.

(5) $W(B_2) \cong D_8$, the dihedral group of order 8.

(6) $W(B_3)$, $|W(B_3)| = 48$.

(7) $W(G_2) \cong D_{12}$, the dihedral group of order 12.

(8) $W(I_2(8)) \cong D_{16}$, the dihedral group of order 16.

(9) $W(A_1) \times W(A_1) \cong D_4$, the dihedral group of order 4.

(10) $W(A_1) \times W(A_2)$, $|W(A_1) \times W(A_2)| = 12$.

(11) $W(A_2) \times W(A_2)$, $|W(A_2) \times W(A_2)| = 36$.

(1) $W = W(A_1) = < w_1 : w_1{}^2 = 1 > (\cong S_2)$

J	$\{w : w \in Y_J\}$	$\|w\|$	$K : w^{-1} \in Y_K$
$\emptyset$	$\{w_1\}$	2	$\emptyset$
$\{w_1\}$	$\{1\}$	1	$\{w_1\}$

Conjugacy classes:

$C_1 = \{1\}, \quad C_2 = \{w_1\}.$

---

(2) $W = W(A_2) = < w_1, w_2 : w_1{}^2 = w_2{}^2 = (w_1 w_2)^3 = 1 > (\cong S_3)$

J	$\{w : w \in Y_J\}$	$\|w\|$	$K : w^{-1} \in Y_K$
$\emptyset$	$\{w_1 w_2 w_1\}$	2	$\emptyset$
$\{w_1\}$	$\left\{\begin{matrix} w_2 \\ w_1 w_2 \end{matrix}\right\}$	2 3	$\{w_1\}$ $\{w_2\}$
$\{w_2\}$	$\left\{\begin{matrix} w_1 \\ w_2 w_1 \end{matrix}\right\}$	2 3	$\{w_2\}$ $\{w_1\}$
$\{w_1, w_2\}$	$\{1\}$	1	$\{w_1, w_2\}$

Conjugacy classes:

$C_1 = \{1\}, \quad C_2 = \{w_1, w_2, w_1 w_2 w_1\}, \quad C_3 = \{w_1 w_2, w_2 w_1\}.$

---

(3) $\underline{W = W(A_3) = \langle w_1, w_2, w_3 : w_1{}^2 = w_2{}^2 = w_3{}^2 = (w_1w_2)^3 = (w_1w_3)^2}$
$\underline{= (w_2w_3)^3 = 1 \rangle \ (\cong S_4)}$

J	$\{w : w \in Y_J\}$	$\lvert w\rvert$	$K : w^{-1} \in Y_K$
$\emptyset$	$\{w_1w_2w_1w_3w_2w_1\}$	2	$\emptyset$
$\{w_1\}$	$w_2w_3w_2$	2	$\{w_1\}$
	$w_1w_2w_3w_2$	3	$\{w_2\}$
	$w_2w_1w_2w_3w_2$	4	$\{w_3\}$
$\{w_2\}$	w_1w_3	2	$\{w_2\}$
	$w_2w_1w_3$	4	$\{w_1, w_3\}$
	$w_1w_2w_1w_3$	3	$\{w_3\}$
	$w_3w_2w_1w_3$	3	$\{w_1\}$
	$w_1w_3w_2w_1w_3$	2	$\{w_2\}$
$\{w_3\}$	$w_1w_2w_1$	2	$\{w_3\}$
	$w_3w_1w_2w_1$	3	$\{w_2\}$
	$w_2w_3w_1w_2w_1$	4	$\{w_1\}$
$\{w_1, w_2\}$	w_3	2	$\{w_1, w_2\}$
	w_2w_3	3	$\{w_1, w_3\}$
	$w_1w_2w_3$	4	$\{w_2, w_3\}$
$\{w_1, w_3\}$	w_2	2	$\{w_1, w_3\}$
	w_1w_2	3	$\{w_2, w_3\}$
	w_3w_2	3	$\{w_1, w_2\}$
	$w_1w_3w_2$	4	$\{w_2\}$
	$w_2w_1w_3w_2$	2	$\{w_1, w_3\}$
$\{w_2, w_3\}$	w_1	2	$\{w_2, w_3\}$
	w_2w_1	3	$\{w_1, w_3\}$
	$w_3w_2w_1$	4	$\{w_1, w_2\}$
$\{w_1, w_2, w_3\}$	$\{1\}$	1	$\{w_1, w_2, w_3\}$

Conjugacy classes:

$$C_1 = \{1\}, \quad C_2 = \left\{\begin{array}{r} w_1 \\ w_2 \\ w_3 \\ w_1w_2w_1 \\ w_2w_3w_2 \\ w_1w_2w_3w_2w_1 \end{array}\right\}, \quad C_3 = \left\{\begin{array}{r} w_1w_3 \\ w_2w_1w_3w_2 \\ w_1w_2w_1w_3w_2w_1 \end{array}\right\}$$

$$C_4 = \left\{\begin{array}{r} w_1w_2 \\ w_2w_1 \\ w_2w_3 \\ w_3w_2 \\ w_1w_2w_3w_2 \\ w_2w_3w_2w_1 \\ w_1w_2w_1w_3 \\ w_1w_3w_2w_1 \end{array}\right\}, \quad C_5 = \left\{\begin{array}{r} w_1w_2w_3 \\ w_3w_1w_2 \\ w_2w_3w_1 \\ w_3w_2w_1 \\ w_2w_1w_2w_3w_2 \\ w_2w_3w_1w_2w_1 \end{array}\right\}$$

(4) $\underline{W = W(A_4) = \langle w_1, w_2, w_3, w_4 : w_i{}^2 = 1 \text{ for } 1 \leqslant i \leqslant 4,\ (w_1w_2)^3 = 1,}$

$\underline{(w_1w_3)^2 = (w_1w_4)^2 = (w_2w_4)^2 = 1,}$

$\underline{(w_2w_3)^3 = (w_3w_4)^3 = 1 \rangle}$

J	$\{w : w \in Y_J\}$	$\lvert w \rvert$	$K : w^{-1} \in Y_K$
$\emptyset$	$\{w_1w_2w_3w_4w_1w_2w_3w_1w_2w_1\}$	2	$\emptyset$
$\{w_1\}$	$w_2w_3w_2w_4w_3w_2$	2	$\{w_1\}$
	$w_1w_2w_3w_4w_2w_3w_2$	6	$\{w_2\}$
	$w_1w_2w_3w_4w_1w_2w_3w_2$	5	$\{w_3\}$
	$w_1w_2w_3w_4w_1w_2w_3w_1w_2$	4	$\{w_4\}$
$\{w_2\}$	$w_1w_3w_4w_3$	2	$\{w_2\}$
	$w_2w_1w_3w_4w_3$	4	$\{w_1, w_3\}$
	$w_1w_2w_1w_3w_4w_3$	3	$\{w_3\}$
	$w_2w_3w_2w_1w_4w_3$	5	$\{w_1, w_4\}$
	$w_1w_2w_3w_4w_2w_1w_3$	4	$\{w_2, w_4\}$
	$w_2w_3w_2w_4w_3w_2w_1$	6	$\{w_1\}$
	$w_1w_2w_3w_4w_1w_2w_3w_1$	5	$\{w_4\}$
	$w_1w_2w_3w_4w_2w_3w_2w_1$	2	$\{w_2\}$
	$w_1w_2w_3w_4w_1w_2w_3w_2w_1$	6	$\{w_3\}$
$\{w_3\}$	$w_1w_2w_1w_4$	2	$\{w_3\}$
	$w_1w_3w_2w_4w_1$	4	$\{w_2, w_4\}$
	$w_2w_1w_3w_2w_1w_4$	5	$\{w_1, w_4\}$
	$w_1w_3w_4w_3w_2w_1$	3	$\{w_2\}$
	$w_1w_2w_3w_4w_1w_2w_1$	6	$\{w_4\}$
	$w_2w_1w_3w_4w_3w_2w_1$	4	$\{w_1, w_3\}$
	$w_1w_2w_3w_4w_1w_3w_2w_1$	2	$\{w_3\}$
	$w_2w_3w_4w_2w_1w_3w_2w_1$	5	$\{w_1\}$
	$w_1w_2w_3w_4w_2w_1w_3w_2w_1$	6	$\{w_2\}$

$W(A_4)$ continued.

J	$\{w : w \in Y_J\}$	$\lvert w\rvert$	$K : w^{-1} \in Y_K$
$\{w_4\}$	$w_1w_2w_1w_3w_2w_1$	2	$\{w_4\}$
	$w_1w_2w_4w_3w_1w_2w_1$	6	$\{w_3\}$
	$w_1w_3w_2w_1w_4w_3w_2w_1$	5	$\{w_2\}$
	$w_2w_1w_3w_2w_4w_3w_1w_2w_1$	4	$\{w_1\}$
$\{w_1, w_2\}$	$w_3w_4w_3$	2	$\{w_1, w_2\}$
	$w_2w_3w_4w_3$	3	$\{w_1, w_3\}$
	$w_1w_2w_3w_4w_3$	4	$\{w_2, w_3\}$
	$w_2w_3w_2w_4w_3$	4	$\{w_1, w_4\}$
	$w_1w_2w_3w_4w_2w_3$	5	$\{w_2, w_4\}$
	$w_1w_2w_3w_4w_1w_2w_3$	6	$\{w_3, w_4\}$
$\{w_2, w_3\}$	w_1w_4	2	$\{w_2, w_3\}$
	$w_2w_1w_4$	6	$\{w_1, w_3\}$
	$w_1w_3w_4$	6	$\{w_2, w_4\}$
	$w_2w_1w_3w_4$	5	$\{w_1, w_3, w_4\}$
	$w_3w_4w_2w_1$	5	$\{w_1, w_2, w_4\}$
	$w_1w_2w_1w_3w_4$	4	$\{w_3, w_4\}$
	$w_2w_3w_2w_1w_4$	4	$\{w_1, w_4\}$
	$w_3w_4w_3w_2w_1$	4	$\{w_1, w_2\}$
	$w_1w_2w_3w_4w_2w_1$	3	$\{w_2, w_4\}$
	$w_2w_3w_4w_3w_2w_1$	3	$\{w_1, w_3\}$
	$w_1w_2w_3w_4w_3w_2w_1$	2	$\{w_2, w_3\}$

W(A_4) continued.

J	$\{w : w \in Y_J\}$	$\lvert w \rvert$	$K : w^{-1} \in Y_K$
$\{w_3, w_4\}$	$w_1w_2w_1$	2	$\{w_3, w_4\}$
	$w_1w_3w_2w_1$	3	$\{w_2, w_4\}$
	$w_2w_1w_3w_2w_1$	4	$\{w_1, w_4\}$
	$w_1w_4w_3w_2w_1$	4	$\{w_2, w_3\}$
	$w_2w_1w_4w_3w_2w_1$	5	$\{w_1, w_3\}$
	$w_3w_4w_2w_1w_3w_2w_1$	6	$\{w_1, w_2\}$
$\{w_1, w_3\}$	w_2w_4	2	$\{w_1, w_3\}$
	$w_1w_2w_4$	6	$\{w_2, w_3\}$
	$w_3w_4w_2$	4	$\{w_1, w_2, w_4\}$
	$w_1w_3w_2w_4$	5	$\{w_2, w_4\}$
	$w_2w_3w_2w_4$	3	$\{w_1, w_4\}$
	$w_3w_4w_3w_2$	3	$\{w_1, w_2\}$
	$w_1w_2w_3w_4w_2$	4	$\{w_2, w_4\}$
	$w_2w_1w_3w_2w_4$	6	$\{w_1, w_3, w_4\}$
	$w_1w_3w_4w_3w_2$	4	$\{w_2\}$
	$w_2w_3w_4w_3w_2$	2	$\{w_1, w_3\}$
	$w_1w_2w_1w_3w_2w_4$	5	$\{w_3, w_4\}$
	$w_1w_2w_3w_4w_3w_2$	3	$\{w_2, w_3\}$
	$w_2w_1w_3w_4w_3w_2$	2	$\{w_1, w_3\}$
	$w_1w_2w_3w_4w_1w_3w_2$	4	$\{w_3\}$
	$w_2w_3w_2w_1w_4w_3w_2$	6	$\{w_1, w_4\}$
	$w_1w_2w_3w_4w_2w_1w_3w_2$	5	$\{w_2, w_4\}$

$W(A_4)$ continued.

J	$\{w : w \in Y_J\}$	$\lvert w\rvert$	$K : w^{-1} \in Y_K$
$\{w_2, w_4\}$	w_1w_3	2	$\{w_2, w_4\}$
	$w_2w_1w_3$	4	$\{w_1, w_3, w_4\}$
	$w_1w_4w_3$	6	$\{w_2, w_3\}$
	$w_1w_2w_1w_3$	3	$\{w_3, w_4\}$
	$w_2w_1w_4w_3$	5	$\{w_1, w_3\}$
	$w_2w_3w_2w_1$	3	$\{w_1, w_4\}$
	$w_1w_2w_1w_4w_3$	4	$\{w_3\}$
	$w_1w_2w_3w_2w_1$	2	$\{w_2, w_4\}$
	$w_2w_4w_3w_2w_1$	4	$\{w_1, w_3\}$
	$w_3w_4w_2w_1w_3$	6	$\{w_1, w_2, w_4\}$
	$w_1w_2w_4w_3w_2w_1$	3	$\{w_2, w_3\}$
	$w_1w_3w_2w_4w_3w_1$	2	$\{w_2, w_4\}$
	$w_3w_2w_4w_3w_2w_1$	5	$\{w_1, w_2\}$
	$w_2w_1w_3w_2w_4w_3w_1$	6	$\{w_1, w_4\}$
	$w_1w_3w_2w_4w_3w_2w_1$	4	$\{w_2\}$
	$w_2w_1w_3w_2w_4w_3w_2w_1$	5	$\{w_1, w_3\}$
$\{w_1, w_4\}$	$w_2w_3w_2$	2	$\{w_1, w_4\}$
	$w_1w_2w_3w_2$	3	$\{w_2, w_4\}$
	$w_2w_4w_3w_2$	3	$\{w_1, w_3\}$
	$w_1w_2w_1w_3w_2$	4	$\{w_3, w_4\}$
	$w_1w_2w_4w_3w_2$	4	$\{w_2, w_3\}$
	$w_3w_4w_3w_2w_3$	4	$\{w_1, w_2\}$
	$w_1w_2w_1w_4w_3w_2$	5	$\{w_3\}$
	$w_1w_3w_2w_4w_3w_2$	5	$\{w_2\}$
	$w_2w_1w_3w_2w_4w_3w_2$	6	$\{w_1, w_3\}$
	$w_1w_3w_2w_1w_4w_3w_2$	6	$\{w_2, w_4\}$
	$w_2w_1w_3w_2w_4w_3w_1w_2$	2	$\{w_1, w_4\}$

$W(A_4)$ continued.

J	$\{w : w \in Y_J\}$	$\lvert w\rvert$	$K : w^{-1} \in Y_K$
$\{w_1, w_2, w_3\}$	w_4	2	$\{w_1, w_2, w_3\}$
	w_3w_4	3	$\{w_1, w_2, w_4\}$
	$w_2w_3w_4$	4	$\{w_1, w_3, w_4\}$
	$w_1w_2w_3w_4$	5	$\{w_2, w_3, w_4\}$
$\{w_1, w_2, w_4\}$	w_3	2	$\{w_1, w_2, w_4\}$
	w_2w_3	3	$\{w_1, w_3, w_4\}$
	w_4w_3	3	$\{w_1, w_2, w_3\}$
	$w_1w_2w_3$	4	$\{w_2, w_3, w_4\}$
	$w_2w_4w_3$	4	$\{w_1, w_3\}$
	$w_1w_2w_4w_3$	5	$\{w_2, w_3\}$
	$w_3w_4w_2w_3$	2	$\{w_1, w_2, w_4\}$
	$w_1w_3w_2w_4w_3$	6	$\{w_2, w_4\}$
	$w_2w_1w_3w_2w_4w_3$	5	$\{w_1, w_3, w_4\}$
$\{w_1, w_3, w_4\}$	w_2	2	$\{w_1, w_3, w_4\}$
	w_1w_2	3	$\{w_2, w_3, w_4\}$
	w_3w_2	3	$\{w_1, w_2, w_4\}$
	$w_1w_3w_2$	4	$\{w_2, w_4\}$
	$w_4w_3w_2$	4	$\{w_1, w_2, w_3\}$
	$w_2w_1w_3w_2$	2	$\{w_1, w_3, w_4\}$
	$w_1w_4w_3w_2$	5	$\{w_2, w_3\}$
	$w_2w_1w_4w_3w_2$	6	$\{w_1, w_3\}$
	$w_3w_2w_1w_4w_3w_2$	5	$\{w_1, w_2, w_4\}$
$\{w_2, w_3, w_4\}$	w_1	2	$\{w_2, w_3, w_4\}$
	w_2w_1	3	$\{w_1, w_3, w_4\}$
	$w_3w_2w_1$	4	$\{w_1, w_2, w_4\}$
	$w_4w_3w_2w_1$	5	$\{w_1, w_2, w_3\}$
$\{w_1, w_2, w_3, w_4\}$	$\{1\}$	1	$\{w_1, w_2, w_3, w_4\}$

$W(A_4)$ <u>Conjugacy classes</u>:

$$C_1 = \{1\},\quad C_2 = \left\{\begin{array}{r} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_1w_2w_1 \\ w_2w_3w_2 \\ w_3w_4w_3 \\ w_1w_2w_3w_2w_1 \\ w_2w_3w_4w_3w_2 \\ w_1w_2w_3w_4w_3w_2w_1 \end{array}\right\}\qquad C_3 = \left\{\begin{array}{r} w_1w_2 \\ w_2w_1 \\ w_2w_3 \\ w_3w_2 \\ w_3w_4 \\ w_4w_3 \\ w_2w_3w_2w_1 \\ w_1w_2w_1w_3 \\ w_3w_4w_3w_2 \\ w_2w_4w_3w_2 \\ w_2w_3w_4w_3 \\ w_2w_3w_2w_4 \\ w_1w_3w_2w_1 \\ w_1w_2w_3w_2 \\ w_1w_2w_1w_3w_4w_3 \\ w_2w_3w_4w_3w_2w_1 \\ w_1w_3w_4w_3w_2w_1 \\ w_1w_2w_4w_3w_2w_1 \\ w_1w_2w_3w_4w_2w_1 \\ w_1w_2w_3w_4w_3w_2 \end{array}\right\}$$

$$C_4 = \left\{\begin{array}{r} w_1w_3 \\ w_1w_4 \\ w_2w_4 \\ w_2w_1w_3w_2 \\ w_1w_3w_4w_3 \\ w_1w_2w_1w_4 \\ w_3w_4w_2w_3 \\ w_1w_2w_1w_3w_2w_1 \\ w_2w_1w_3w_4w_3w_2 \\ w_1w_3w_2w_4w_3w_1 \\ w_2w_3w_2w_4w_3w_2 \\ w_1w_2w_3w_4w_1w_3w_2w_1 \\ w_1w_2w_3w_4w_2w_3w_2w_1 \\ w_2w_1w_3w_2w_4w_3w_1w_2 \\ w_1w_2w_3w_4w_1w_2w_3w_1w_2w_1 \end{array}\right\}$$

$W(A_4)$ Conjugacy classes:

$$C_5 = \left\{\begin{array}{r} w_1w_2w_3 \\ w_2w_1w_3 \\ w_1w_3w_2 \\ w_2w_3w_4 \\ w_3w_2w_1 \\ w_2w_4w_3 \\ w_3w_4w_2 \\ w_4w_3w_2 \\ w_1w_2w_1w_3w_2 \\ w_1w_2w_1w_3w_4 \\ w_1w_2w_1w_4w_3 \\ w_1w_2w_3w_4w_3 \\ w_1w_2w_3w_4w_2 \\ w_1w_2w_4w_3w_2 \\ w_2w_1w_3w_2w_1 \\ w_2w_1w_3w_4w_3 \\ w_1w_3w_2w_4w_1 \\ w_1w_3w_4w_3w_2 \\ w_2w_3w_2w_1w_4 \\ w_2w_3w_2w_4w_3 \\ w_1w_4w_3w_2w_1 \\ w_2w_4w_3w_2w_1 \\ w_3w_4w_3w_2w_1 \\ w_3w_4w_3w_2w_3 \\ w_1w_2w_3w_4w_1w_3w_2 \\ w_1w_2w_3w_4w_2w_1w_3 \\ w_2w_1w_3w_4w_3w_2w_1 \quad w_1w_2w_3w_4w_1w_2w_3w_1w_2 \\ w_1w_3w_2w_4w_3w_2w_1 \quad w_2w_1w_3w_2w_4w_3w_1w_2w_1 \end{array}\right\}$$

$W(A_4)$ Conjugacy classes:

$$C_6 = \left\{\begin{array}{r}
w_1w_2w_4\\
w_2w_1w_4\\
w_1w_3w_4\\
w_1w_4w_3\\
w_2w_1w_3w_2w_4\\
w_2w_1w_4w_3w_2\\
w_1w_3w_2w_4w_3\\
w_3w_4w_2w_1w_3\\
w_1w_2w_3w_4w_1w_2w_3\\
w_1w_2w_3w_4w_1w_2w_1\\
w_2w_1w_3w_2w_4w_3w_1\\
w_2w_1w_3w_2w_4w_3w_2\\
w_1w_3w_2w_1w_4w_3w_2\\
w_1w_2w_4w_3w_1w_2w_1\\
w_1w_2w_3w_4w_2w_3w_2\\
w_2w_3w_2w_1w_4w_3w_2\\
w_3w_4w_2w_1w_3w_2w_1\\
w_2w_3w_2w_4w_3w_2w_1\\
w_1w_2w_3w_4w_1w_2w_3w_2w_1\\
w_1w_2w_3w_4w_2w_1w_3w_2w_1
\end{array}\right\}
\qquad
C_7 = \left\{\begin{array}{r}
w_1w_2w_3w_4\\
w_1w_2w_4w_3\\
w_2w_1w_3w_4\\
w_2w_1w_4w_3\\
w_1w_3w_2w_4\\
w_1w_4w_3w_2\\
w_3w_4w_2w_1\\
w_4w_3w_2w_1\\
w_1w_2w_1w_3w_2w_4\\
w_1w_2w_1w_4w_3w_2\\
w_1w_2w_3w_4w_2w_3\\
w_2w_1w_3w_2w_1w_4\\
w_2w_1w_3w_2w_4w_3\\
w_2w_1w_4w_3w_2w_1\\
w_1w_3w_2w_4w_3w_2\\
w_2w_3w_2w_1w_4w_3\\
w_3w_2w_4w_3w_2w_1\\
w_3w_2w_1w_4w_3w_2\\
w_1w_2w_3w_4w_1w_2w_3w_1\\
w_1w_2w_3w_4w_1w_2w_3w_2\\
w_1w_2w_3w_4w_2w_1w_3w_2\\
w_1w_3w_2w_1w_4w_3w_2w_1\\
w_2w_3w_4w_2w_1w_3w_2w_1\\
w_2w_1w_3w_2w_4w_3w_2w_1
\end{array}\right\}$$

(5) $W = W(B_2) = \langle w_1, w_2 : w_1^{\,2} = w_2^{\,2} = (w_1w_2)^4 = 1 \rangle$.

J	$\{w : w \in Y_J\}$	$\lvert w \rvert$	$K : w^{-1} \in Y_K$
$\emptyset$	$\{w_1w_2w_1w_2\}$	2	$\emptyset$
$\{w_1\}$	w_2	2	$\{w_1\}$
	w_1w_2	4	$\{w_2\}$
	$w_2w_1w_2$	2	$\{w_1\}$
$\{w_2\}$	w_1	2	$\{w_2\}$
	w_2w_1	4	$\{w_1\}$
	$w_1w_2w_1$	2	$\{w_2\}$
$\{w_1, w_2\}$	$\{1\}$	1	$\{w_1, w_2\}$

Conjugacy classes:

$$C_1 = \{1\}, \quad C_2 = \left\{\begin{matrix} w_2 \\ w_1w_2w_1 \end{matrix}\right\}, \quad C_3 = \left\{\begin{matrix} w_1 \\ w_2w_1w_2 \end{matrix}\right\}, \quad C_4 = \left\{\begin{matrix} w_1w_2 \\ w_2w_1 \end{matrix}\right\}$$

$$C_5 = \{w_1w_2w_1w_2\}$$

(6) $\underline{W = W(B_3) = \langle w_1, w_2, w_3 : w_1^{\,2} = w_2^{\,2} = w_3^{\,2} = 1,}$

$$\underline{(w_1w_2)^4 = (w_1w_3)^2 = (w_2w_3)^3 = 1\rangle.}$$

J	$\{w : w \in Y_J\}$	$\lvert w\rvert$	$K : w^{-1} \in Y_K$
$\emptyset$	$\{w_1w_2w_1w_2w_3w_2w_1w_2w_3\}$	2	$\emptyset$
$\{w_1\}$	$w_2w_3w_2$	2	$\{w_1\}$
	$w_1w_2w_3w_2$	4	$\{w_2\}$
	$w_2w_1w_2w_3w_2$	6	$\{w_1, w_3\}$
	$w_2w_1w_2w_3w_1w_2$	3	$\{w_3\}$
	$w_3w_2w_1w_2w_3w_2$	2	$\{w_1\}$
	$w_1w_3w_2w_1w_2w_3w_2$	4	$\{w_2\}$
	$w_3w_2w_1w_2w_3w_2w_1w_2$	2	$\{w_1\}$
$\{w_2\}$	w_1w_3	2	$\{w_2\}$
	$w_2w_1w_3$	6	$\{w_1, w_3\}$
	$w_2w_3w_2w_1$	4	$\{w_1\}$
	$w_1w_2w_1w_3$	3	$\{w_2, w_3\}$
	$w_1w_2w_3w_2w_1$	2	$\{w_2\}$
	$w_2w_1w_2w_1w_3$	4	$\{w_3\}$
	$w_2w_1w_3w_2w_1w_3$	3	$\{w_1, w_3\}$
	$w_3w_2w_1w_2w_3w_1$	2	$\{w_2\}$
	$w_2w_1w_2w_3w_1w_2w_1$	6	$\{w_3\}$
	$w_3w_2w_1w_2w_3w_2w_1$	4	$\{w_1\}$
	$w_3w_2w_1w_2w_3w_1w_2w_1$	2	$\{w_2\}$
$\{w_3\}$	$w_1w_2w_1w_2$	2	$\{w_3\}$
	$w_3w_1w_2w_1w_2$	4	$\{w_2\}$
	$w_2w_3w_1w_2w_1w_2$	3	$\{w_1\}$
	$w_1w_2w_3w_2w_1w_2w_1$	6	$\{w_2\}$
	$w_2w_1w_2w_3w_2w_1w_2w_1$	2	$\{w_3\}$

$W(B_3)$ continued.

J	$\{w : w \quad Y_J\}$	w	$K : w^{-1} \quad Y_K$
$\{w_1, w_2\}$	w_3	2	$\{w_1, w_2\}$
	w_2w_3	3	$\{w_1, w_3\}$
	$w_1w_2w_3$	6	$\{w_2, w_3\}$
	$w_2w_1w_2w_3$	4	$\{w_1, w_3\}$
	$w_3w_2w_1w_2w_3$	2	$\{w_1, w_2\}$
$\{w_1, w_3\}$	w_2	2	$\{w_1, w_3\}$
	w_1w_2	4	$\{w_2, w_3\}$
	w_3w_2	3	$\{w_1, w_2\}$
	$w_1w_3w_2$	6	$\{w_2\}$
	$w_2w_1w_2$	2	$\{w_1, w_3\}$
	$w_3w_2w_1w_2$	4	$\{w_1, w_2\}$
	$w_2w_3w_1w_2$	2	$\{w_1, w_3\}$
	$w_2w_3w_2w_1w_2$	6	$\{w_1\}$
	$w_1w_2w_3w_1w_2$	4	$\{w_2, w_3\}$
	$w_1w_2w_3w_2w_1w_2$	3	$\{w_2\}$
	$w_2w_1w_2w_3w_2w_1w_2$	2	$\{w_1, w_3\}$
$\{w_2, w_3\}$	w_1	2	$\{w_2, w_3\}$
	w_2w_1	4	$\{w_1, w_3\}$
	$w_1w_2w_1$	2	$\{w_2, w_3\}$
	$w_3w_2w_1$	6	$\{w_1, w_2\}$
	$w_3w_1w_2w_1$	3	$\{w_2\}$
	$w_2w_3w_1w_2w_1$	4	$\{w_1, w_3\}$
	$w_1w_2w_3w_1w_2w_1$	2	$\{w_2, w_3\}$
$\{w_1, w_2, w_3\}$	$\{1\}$	1	$\{w_1, w_2, w_3\}$

$W(B_3)$ Conjugacy classes:

$$C_1 = \{1\},\quad C_2 = \left\{\begin{array}{r} w_1 \\ w_2w_1w_2 \\ w_3w_2w_1w_2w_3 \end{array}\right\},\quad C_3 = \left\{\begin{array}{r} w_2 \\ w_3 \\ w_1w_2w_1 \\ w_2w_3w_2 \\ w_1w_2w_3w_2w_1 \\ w_2w_1w_2w_3w_2w_1w_2 \end{array}\right\}$$

$$C_4 = \left\{\begin{array}{r} w_1w_3 \\ w_2w_3w_1w_2 \\ w_1w_2w_3w_1w_2w_1 \\ w_3w_2w_1w_2w_3w_2 \\ w_2w_1w_2w_3w_2w_1w_2w_1 \\ w_3w_2w_1w_2w_3w_1w_2w_1 \end{array}\right\}\quad C_5 = \left\{\begin{array}{r} w_1w_2 \\ w_2w_1 \\ w_1w_2w_3w_2 \\ w_2w_3w_2w_1 \\ w_2w_1w_2w_3 \\ w_3w_2w_1w_2 \end{array}\right\}$$

$$C_6 = \left\{\begin{array}{r} w_2w_3 \\ w_3w_2 \\ w_1w_2w_1w_3 \\ w_3w_1w_2w_1 \\ w_2w_3w_1w_2w_1w_2 \\ w_2w_1w_2w_3w_1w_2 \\ w_1w_2w_3w_2w_1w_2 \\ w_2w_1w_2w_3w_2w_1 \end{array}\right\}\quad C_7 = \left\{\begin{array}{r} w_1w_2w_3 \\ w_3w_2w_1 \\ w_2w_1w_3 \\ w_1w_3w_2 \\ w_2w_3w_2w_1w_2 \\ w_2w_1w_2w_3w_2 \\ w_1w_2w_3w_2w_1w_2w_1 \\ w_2w_1w_2w_3w_1w_2w_1 \end{array}\right\}$$

$$C_8 = \left\{\begin{array}{r} w_1w_2w_1w_2 \\ w_3w_2w_1w_2w_3w_1 \\ w_3w_2w_1w_2w_3w_2w_1w_2 \end{array}\right\}\quad C_9 = \left\{\begin{array}{r} w_3w_1w_2w_1w_2 \\ w_2w_1w_2w_1w_3 \\ w_2w_3w_1w_2w_1 \\ w_1w_2w_3w_1w_2 \\ w_1w_3w_2w_1w_2w_3w_2 \\ w_3w_2w_1w_2w_3w_2w_1 \end{array}\right\}$$

$$C_{10} = \{w_1w_2w_1w_2w_3w_2w_1w_2w_3\}$$

(7) $W = W(G_2) = \langle w_1, w_2 : w_1^2 = w_2^2 = (w_1w_2)^6 = 1 \rangle$

J	$\{w : w \in Y_J\}$	$\lvert w \rvert$	$K : w^{-1} \in Y_K$
$\emptyset$	$\{w_1w_2w_1w_2w_1w_2\}$	2	$\emptyset$
$\{w_1\}$	w_2	2	$\{w_1\}$
	w_1w_2	6	$\{w_2\}$
	$w_2w_1w_2$	2	$\{w_1\}$
	$w_1w_2w_1w_2$	4	$\{w_2\}$
	$w_2w_1w_2w_1w_2$	2	$\{w_1\}$
$\{w_2\}$	w_1	2	$\{w_2\}$
	w_2w_1	6	$\{w_1\}$
	$w_1w_2w_1$	2	$\{w_2\}$
	$w_2w_1w_2w_1$	4	$\{w_1\}$
	$w_1w_2w_1w_2w_1$	2	$\{w_2\}$
$\{w_1, w_2\}$	$\{1\}$	1	$\{w_1, w_2\}$

Conjugacy classes:

$$C_1 = \{1\}, \quad C_2 = \left\{\begin{matrix} w_1 \\ w_2w_1w_2 \\ w_1w_2w_1w_2w_1 \end{matrix}\right\}, \quad C_3 = \left\{\begin{matrix} w_2 \\ w_1w_2w_1 \\ w_2w_1w_2w_1w_2 \end{matrix}\right\}, \quad C_4 = \left\{\begin{matrix} w_1w_2 \\ w_2w_1 \end{matrix}\right\}$$

$$C_5 = \left\{\begin{matrix} w_1w_2w_1w_2 \\ w_2w_1w_2w_1 \end{matrix}\right\}, \quad C_6 = \{w_1w_2w_1w_2w_1w_2\}$$

(8) $W = W(I_2(8)) = \langle w_1, w_2 : w_1^2 = w_2^2 = (w_1w_2)^8 = 1 \rangle \quad (\cong D_{16})$

J	$\{w : w \in Y_J\}$	$\lvert w \rvert$	$K : w^{-1} \in Y_K$
$\emptyset$	$\{w_1w_2w_1w_2w_1w_2w_1w_2\}$	2	$\emptyset$
$\{w_1\}$	w_2	2	$\{w_1\}$
	w_1w_2	8	$\{w_2\}$
	$w_2w_1w_2$	2	$\{w_1\}$
	$w_1w_2w_1w_2$	4	$\{w_2\}$
	$w_2w_1w_2w_1w_2$	2	$\{w_1\}$
	$w_1w_2w_1w_2w_1w_2$	8	$\{w_2\}$
	$w_2w_1w_2w_1w_2w_1w_2$	2	$\{w_1\}$
$\{w_2\}$	w_1	2	$\{w_2\}$
	w_2w_1	8	$\{w_1\}$
	$w_1w_2w_1$	2	$\{w_2\}$
	$w_2w_1w_2w_1$	4	$\{w_1\}$
	$w_1w_2w_1w_2w_1$	2	$\{w_2\}$
	$w_2w_1w_2w_1w_2w_1$	8	$\{w_1\}$
	$w_1w_2w_1w_2w_1w_2w_1$	2	$\{w_2\}$
$\{w_1, w_2\}$	$\{1\}$	1	$\{w_1, w_2\}$

Conjugacy classes:

$$C_1 = \{1\}, \quad C_2 = \begin{Bmatrix} w_1 \\ w_2w_1w_2 \\ w_1w_2w_1w_2w_1 \\ w_2w_1w_2w_1w_2w_1w_2 \end{Bmatrix}, \quad C_3 = \begin{Bmatrix} w_2 \\ w_1w_2w_1 \\ w_2w_1w_2w_1w_2 \\ w_1w_2w_1w_2w_1w_2w_1 \end{Bmatrix}$$

$$C_4 = \begin{Bmatrix} w_1w_2 \\ w_2w_1 \end{Bmatrix}$$

$$C_5 = \begin{Bmatrix} w_1w_2w_1w_2 \\ w_2w_1w_2w_1 \end{Bmatrix}, \quad C_6 = \begin{Bmatrix} w_1w_2w_1w_2w_1w_2 \\ w_2w_1w_2w_1w_2w_1 \end{Bmatrix}, \quad C_7 = \{w_1w_2w_1w_2w_1w_2w_1w_2\}$$

(9) $W = W(A_1) \times W(A_1) = \langle w_1, w_2 : w_1^2 = w_2^2 = (w_1w_2)^2 = 1 \rangle$

J	$\{w : w \in Y_J\}$	$\lvert w \rvert$	$K : w^{-1} \in Y_K$
$\emptyset$	$\{w_1w_2\}$	2	$\emptyset$
$\{w_1\}$	$\{w_2\}$	2	$\{w_1\}$
$\{w_2\}$	$\{w_1\}$	2	$\{w_2\}$
$\{w_1, w_2\}$	$\{1\}$	1	$\{w_1, w_2\}$

Conjugacy classes:

$C_1 = \{1\}$, $C_2 = \{w_1\}$, $C_3 = \{w_2\}$, $C_4 = \{w_1w_2\}$.

(10) $\underline{W = W(A_1) \times W(A_2) = \langle w_1, w_2, w_3 : w_1^2 = w_2^2 = w_3^2 = 1,}$

$\underline{(w_1w_2)^2 = (w_1w_3)^2 = (w_2w_3)^3 = 1\rangle}$

J	$\{w : w \in Y_J\}$	$\lvert w\rvert$	$K : w^{-1} \in Y_K$
$\emptyset$	$\{w_1w_2w_3w_2\}$	2	$\emptyset$
$\{w_1\}$	$\{w_2w_3w_2\}$	2	$\{w_1\}$
$\{w_2\}$	w_1w_3	2	$\{w_2\}$
	$w_1w_2w_3$	6	$\{w_3\}$
$\{w_3\}$	w_1w_2	2	$\{w_3\}$
	$w_1w_3w_2$	6	$\{w_2\}$
$\{w_1, w_2\}$	w_3	2	$\{w_1, w_2\}$
	w_2w_3	3	$\{w_1, w_3\}$
$\{w_1, w_3\}$	w_2	2	$\{w_1, w_3\}$
	w_3w_2	3	$\{w_1, w_2\}$
$\{w_2, w_3\}$	$\{w_1\}$	2	$\{w_2, w_3\}$
$\{w_1, w_2, w_3\}$	$\{1\}$	1	$\{w_1, w_2, w_3\}$

<u>Conjugacy</u> <u>classes</u>:

$$C_1 = \{1\}, \quad C_2 = \{w_1\}, \quad C_3 = \left\{\begin{matrix} w_2 \\ w_3 \\ w_2w_3w_2 \end{matrix}\right\}, \quad C_4 = \left\{\begin{matrix} w_1w_2 \\ w_1w_3 \\ w_1w_2w_3w_2 \end{matrix}\right\},$$

$$C_5 = \left\{\begin{matrix} w_2w_3 \\ w_3w_2 \end{matrix}\right\}, \quad C_6 = \left\{\begin{matrix} w_1w_2w_3 \\ w_1w_3w_2 \end{matrix}\right\}.$$

(11) $$\underline{W = W(A_2)\times W(A_2) = \langle w_1,w_2,w_3,w_4 : w_1^{\,2} = w_2^{\,2} = w_3^{\,2} = w_4^{\,2} = 1,}$$
$$\underline{(w_1w_2)^3 = (w_1w_3)^2 = (w_1w_4)^2 = 1,}$$
$$\underline{(w_2w_3)^2 = (w_2w_4)^2 = (w_3w_4)^3 = 1\rangle}$$

J	$\{w : w \in Y_J\}$	$\lvert w\rvert$	$K : w^{-1} \in Y_K$
$\emptyset$	$\{w_1w_2w_1w_3w_4w_3\}$	2	$\emptyset$
$\{w_1\}$	$w_2w_3w_4w_3$	2	$\{w_1\}$
	$w_1w_2w_3w_4w_3$	6	$\{w_2\}$
$\{w_2\}$	$w_1w_3w_4w_3$	2	$\{w_2\}$
	$w_2w_1w_3w_4w_3$	6	$\{w_1\}$
$\{w_3\}$	$w_1w_2w_1w_4$	2	$\{w_3\}$
	$w_3w_4w_1w_2w_1$	6	$\{w_4\}$
$\{w_4\}$	$w_3w_1w_2w_1$	2	$\{w_4\}$
	$w_4w_3w_1w_2w_1$	6	$\{w_3\}$
$\{w_1,w_2\}$	$\{w_3w_4w_3\}$	2	$\{w_1,w_2\}$
$\{w_1,w_3\}$	w_2w_4	2	$\{w_1,w_3\}$
	$w_1w_2w_4$	6	$\{w_2,w_3\}$
	$w_3w_2w_4$	6	$\{w_1,w_4\}$
	$w_1w_3w_2w_4$	3	$\{w_2,w_4\}$
$\{w_1,w_4\}$	w_2w_3	2	$\{w_1,w_4\}$
	$w_1w_2w_3$	6	$\{w_2,w_4\}$
	$w_4w_2w_3$	6	$\{w_1,w_3\}$
	$w_1w_4w_2w_3$	3	$\{w_2,w_3\}$
$\{w_2,w_3\}$	w_1w_4	2	$\{w_2,w_3\}$
	$w_2w_1w_4$	6	$\{w_1,w_3\}$
	$w_3w_1w_4$	6	$\{w_2,w_4\}$
	$w_2w_3w_1w_4$	3	$\{w_1,w_4\}$

$W(A_2) \times W(A_2)$ continued.

J	$\{w : w \in Y_J\}$	$\lvert w\rvert$	$K : w^{-1} \in Y_K$
$\{w_2, w_4\}$	w_1w_3	2	$\{w_2, w_4\}$
	$w_2w_1w_3$	6	$\{w_1, w_4\}$
	$w_4w_1w_3$	6	$\{w_2, w_3\}$
	$w_2w_4w_1w_3$	3	$\{w_1, w_3\}$
$\{w_3, w_4\}$	$\{w_1w_2w_1\}$	2	$\{w_3, w_4\}$
$\{w_1, w_2, w_3\}$	w_4	2	$\{w_1, w_2, w_3\}$
	w_3w_4	3	$\{w_1, w_2, w_4\}$
$\{w_1, w_2, w_4\}$	w_3	2	$\{w_1, w_2, w_4\}$
	w_4w_3	3	$\{w_1, w_2, w_3\}$
$\{w_1, w_3, w_4\}$	w_2	2	$\{w_1, w_3, w_4\}$
	w_1w_2	3	$\{w_2, w_3, w_4\}$
$\{w_2, w_3, w_4\}$	w_1	2	$\{w_2, w_3, w_4\}$
	w_2w_1	3	$\{w_1, w_3, w_4\}$
$\{w_1, w_2, w_3, w_4\}$	$\{1\}$	1	$\{w_1, w_2, w_3, w_4\}$

Conjugacy classes:

$$C_1 = \{1\}, \quad C_2 = \left\{\begin{matrix} w_1 \\ w_2 \\ w_1w_2w_1 \end{matrix}\right\}, \quad C_3 = \left\{\begin{matrix} w_3 \\ w_4 \\ w_3w_4w_3 \end{matrix}\right\}, \quad C_4 = \left\{\begin{matrix} w_1w_2 \\ w_2w_1 \end{matrix}\right\}, \quad C_5 = \left\{\begin{matrix} w_3w_4 \\ w_4w_3 \end{matrix}\right\}$$

$$C_6 = \left\{\begin{matrix} w_1w_3 & \\ w_1w_4 & \\ w_2w_3 & \\ w_2w_4 & \\ w_3w_1w_2w_1 & w_1w_2w_1w_4 \\ w_1w_3w_4w_3 & w_1w_2w_1w_3w_4w_3 \\ w_2w_3w_4w_3 & \end{matrix}\right\}$$

$W(A_2) \times W(A_2)$ continued.

$$C_7 = \left\{\begin{array}{r} w_1w_2w_4 \\ w_1w_2w_3 \\ w_2w_1w_3 \\ w_2w_1w_4 \\ w_1w_2w_3w_4w_3 \\ w_2w_1w_3w_4w_3 \end{array}\right\} \qquad C_8 = \left\{\begin{array}{r} w_4w_2w_3 \\ w_3w_2w_4 \\ w_4w_1w_3 \\ w_3w_2w_4 \\ w_3w_4w_1w_2w_1 \\ w_4w_3w_1w_2w_1 \end{array}\right\}$$

Appendix 3: IDEMPOTENTS IN THE DECOMPOSITIONS OF THE 0-HECKE ALGEBRA.

A direct sum decomposition of the 0-Hecke algebra H over the field K is equivalent to writing the identity element of H as a sum of mutually orthogonal primitive idempotents. Let $1 = \sum_{J \subseteq R} q_J$ and $1 = \sum_{J \subseteq R} p_J$ be the decompositions of 1 corresponding to the decompositions $H = \sum^{\oplus}_{J \subseteq R} Ho_J^{\wedge} e_J$ and $H = \sum^{\oplus}_{J \subseteq R} He_J o_J^{\wedge}$ respectively, where $Hq_J = Ho_J^{\wedge} e_J$ and $Hp_J = He_J o_J^{\wedge}$ for all $J \subseteq R$.

Since $q_J \in Ho_J^{\wedge} e_J$ and $p_J \in He_J o_J^{\wedge}$, there exist elements b_y, $u_y \in K$ such that $q_J = \sum_{y \in Y_J} b_y a_y o_J^{\wedge} e_J$ and $p_J = \sum_{y \in Y_J} u_y a_y e_J o_J^{\wedge}$.

(A3.1) LEMMA: Suppose there exist elements $q_J' \in Ho_J^{\wedge} e_J$ and $p_J' \in He_J o_J^{\wedge}$ such that $1 = \sum_{J \subseteq R} q_J'$ and $1 = \sum_{J \subseteq R} p_J'$. Then $\{q_J' : J \subseteq R\}$ and $\{p_J' : J \subseteq R\}$ are both sets of mutually orthogonal primitive idempotents, and $Hq_J' = Ho_J^{\wedge} e_J$ and $Hp_J' = He_J o_J^{\wedge}$ for all $J \subseteq R$.

Proof: $q_J' = q_J' 1 = \sum_{L \subseteq R} q_J' q_L'$. Then for all $L \subseteq R$, $q_J' q_L' \in Ho_L^{\wedge} e_L$, and as $\sum_{L \subseteq R} Ho_L^{\wedge} e_L$ is direct, we have $q_J' q_L' = 0$ if $J \neq L$, and $q_J' q_J' = q_J'$. Hence $\{q_J' : J \subseteq R\}$ is a set of mutually orthogonal idempotents, and $Hq_J' \leqslant Ho_J^{\wedge} e_J$ for all $J \subseteq R$. Since $1 = \sum_{J \subseteq R} q_J'$, we have $H = \sum^{\oplus}_{J \subseteq R} Hq_J' \leqslant \sum^{\oplus}_{J \subseteq R} Ho_J^{\wedge} e_J = H$. Hence $Hq_J' = Ho_J^{\wedge} e_J$ for all

$J \subseteq R$, and as $Ho_J^{\wedge}e_J$ is an indecomposable H-module for all $J \subseteq R$, q_J' is primitive for all $J \subseteq R$.

Similarly for $\{p_J' : J \subseteq R\}$.

(A3.2) <u>THEOREM</u>: Let H be the 0-Hecke algebra of type (W,R) where W is the dihedral group of order 2n. For each $J \subseteq R$, let

$$q_J = \sum_{\substack{y \in Y_J \\ y^2 = 1}} (-1)^{l(y)} a_y o_J^{\wedge} e_J.$$

Then $\{q_J : J \subseteq R\}$ is a set of mutually orthogonal primitive idempotents, with $Hq_J = Ho_J^{\wedge}e_J$ for all $J \subseteq R$, and

$$H = \sum_{J \subseteq R}^{\oplus} Hq_J.$$

<u>Proof</u>: Since $q_J \in Ho_J^{\wedge}e_J$ for each $J \subseteq R$, it is sufficient to prove that $1 = \sum_{J \subseteq R} q_J$.

(a) Suppose $n = 2m$ is even. Then

$$q_\emptyset = a_{(w_1w_2w_1..)_{2m}},$$

$$q_{\{w_1\}} = -(a_{w_2} + a_{w_2w_1w_2} + .. + a_{(w_2w_1w_2..)_{2m-1}})(1+a_{w_1})$$

$$= -\sum_{k=1}^{2m} a_{(w_2w_1w_2..)_k}$$

$$q_{\{w_2\}} = -\sum_{k=1}^{2m} a_{(w_1w_2w_1..)_k}$$

$$q_{\{w_1,w_2\}} = \sum_{w \in W} a_w$$

Let $S = q_\emptyset + q_{\{w_1\}} + q_{\{w_2\}} + q_{\{w_1,w_2\}}$. Then by inspection 1 occurs in S with coefficient 1, and for all $w \in W$, $w \neq 1$, a_w occurs with zero coefficient. Hence $S = 1$, and the result is true.

(b) Suppose $n = 2m+1$ is odd. Then

$$q_{\emptyset} = -a_{(w_1w_2w_1..)_{2m+1}},$$

$$q_{\{w_1\}} = -\sum_{k=1}^{2m} a_{(w_2w_1w_2..)_k},$$

$$q_{\{w_2\}} = -\sum_{k=1}^{2m} a_{(w_1w_2w_1..)_k},$$

$$q_{\{w_1,w_2\}} = \sum_{w \in W} a_w.$$

Clearly, $q_{\emptyset} + q_{\{w_1\}} + q_{\{w_2\}} + q_{\{w_1,w_2\}} = 1$, and the result is true.

(A3.3) <u>THEOREM</u>: Let H be the 0-Hecke algebra of type (W,R) where W is a dihedral group of order 2n. Let $p_{\emptyset} = (-1)^{l(w_o)} a_{w_o}$, $p_{\{w_1,w_2\}} = \sum_{w \in W} a_w$, and for each of $J = \{w_1\}, \{w_2\}$ let

$$p_J = (-1) \sum_{y \in Y_J} n_y a_y e_J o_J$$

where for all $y \in Y_J$, $n_y = \left[\frac{l(y)+1}{2}\right]$, where if $x \in Q$, we denote by $[x]$ the largest rational integer which is less than or equal to x. Then $\{p_J : J \subseteq R\}$ is a set of mutually orthogonal primitive idempotents, with $Hp_J = He_J o_{\hat{J}}$ for all $J \subseteq R$, and $H = \sum^{\oplus}_{J \subseteq R} Hp_J$.

<u>Proof</u>: Since $p_J \in He_J o_{\hat{J}}$ for each $J \subseteq R$, it is sufficient to prove that $1 = \sum_{J \subseteq R} p_J$.

(a) Suppose $n = 2m$ is even; then

$$p_{\emptyset} = a_{(w_1w_2w_1..)_{2m}},$$

$$p_{\{w_1\}} = (-1) \sum_{y \in Y_{\{w_1\}}} n_y a_y (1 + a_{w_1})(-a_{w_2})$$

$$= \sum_{k=1}^{2m-1} \left[\frac{k+1}{2}\right] a_{(..w_2w_1w_2)_k} a_{w_2}$$

$$+ \sum_{k=1}^{2m-1} \left[\frac{k+1}{2}\right] a_{(..w_2w_1w_2)_k} a_{w_1} a_{w_2}$$

$$= -\sum_{k=1}^{2m-1} \left[\frac{k+1}{2}\right] a_{(..w_2w_1w_2)_k} + \sum_{k=3}^{2m-1} \left[\frac{k-1}{2}\right] a_{(..w_2w_1w_2)_k}$$

$$+ \left[\frac{2m-1}{2}\right] a_{(..w_2w_1w_2)_{2m}} - \left[\frac{2m}{2}\right] a_{(..w_2w_1w_2)_{2m}}.$$

Now if $k = 1$ or 2, $a_{(..w_2w_1w_2)_k}$ occurs in the expression for $p_{\{w_1\}}$ with coefficient (-1). Since $\left[\frac{k+1}{2}\right] - \left[\frac{k-1}{2}\right] = 1$ for all k, $3 \leq k \leq 2m-1$, the coefficient of $a_{(..w_2w_1w_2)_k}$ for all k, $3 \leq k \leq 2m-1$, in the expression for $p_{\{w_1\}}$ is (-1). Finally, we have that $\left[\frac{2m-1}{2}\right] - \left[\frac{2m}{2}\right] = -1$, and hence

$$p_{\{w_1\}} = -\sum_{k=1}^{2m} a_{(..w_2w_1w_2)_k}, \text{ and similarly}$$

$$p_{\{w_2\}} = -\sum_{k=1}^{2m} a_{(..w_1w_2w_1)_k}.$$

Now clearly we have that $p_\emptyset + p_{\{w_1\}} + p_{\{w_2\}} + p_{\{w_1,w_2\}} = 1$, and hence the result.

(b) Suppose $n = 2m+1$ is odd; then

$$p_\emptyset = -a_{(w_1w_2w_1..)_{2m+1}},$$

$$p_{\{w_1\}} = \sum_{k=1}^{2m} \left[\frac{k+1}{2}\right] a_{(..w_2w_1w_2)_k} a_{w_2}$$

$$+ \sum_{k=1}^{2m} \left[\frac{k+1}{2}\right] a_{(..w_2w_1w_2)_k} a_{w_1} a_{w_2}$$

$$\text{Thus, } p_{\{w_1\}} = -\sum_{k=1}^{2m}\left[\frac{k+1}{2}\right]a_{(..w_2w_1w_2)_k} + \sum_{k=3}^{2m}\left[\frac{k-1}{2}\right]a_{(..w_2w_1w_2)_k}$$

$$+\left[\frac{2m}{2}\right]a_{(..w_2w_1w_2)_{2m+1}} - \left[\frac{2m+1}{2}\right]a_{(..w_2w_1w_2)_{2m+1}}$$

$$= -\sum_{k=1}^{2m} a_{(..w_2w_1w_2)_k}$$

since $\left[\frac{k+1}{2}\right] = 1$ if $k = 1$ or 2, $\left[\frac{k+1}{2}\right] - \left[\frac{k-1}{2}\right] = 1$ for all k, $3 \leq k \leq 2m$, and $\left[\frac{2m}{2}\right] - \left[\frac{2m+1}{2}\right] = 0$. Similarly,

$$p_{\{w_2\}} = -\sum_{k=1}^{2m} a_{(..w_1w_2w_1)_k},$$

and then we have that $p_\emptyset + p_{\{w_1\}} + p_{\{w_2\}} + p_{\{w_1,w_2\}} = 1$, and hence the result.

<u>EXAMPLE</u>: $W = W(G_2)$, the dihedral group of order 12.

Let $R = \{w_1, w_2\}$; then

$$q_\emptyset = a_{w_1w_2w_1w_2w_1w_2}$$

$$q_{\{w_1\}} = -a_{w_2}(1 + a_{w_1}) - a_{w_2w_1w_2}(1 + a_{w_1}) - a_{w_2w_1w_2w_1w_2}(1 + a_{w_1})$$

$$q_{\{w_2\}} = -a_{w_1}(1 + a_{w_2}) - a_{w_1w_2w_1}(1 + a_{w_2}) - a_{w_1w_2w_1w_2w_1}(1 + a_{w_2})$$

$$q_{\{w_1,w_2\}} = (1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1}) \times (1 + a_{w_2}).$$

$$p_\emptyset = a_{w_1w_2w_1w_2w_1w_2}$$

$$p_{\{w_1\}} = (a_{w_2} + a_{w_1w_2} + 2a_{w_2w_1w_2} + 2a_{w_1w_2w_1w_2} + 33a_{w_2w_1w_2w_1w_2})$$

$$\times\ (1 + a_{w_1})a_{w_2}.$$

$$p_{\{w_2\}} = (a_{w_1} + a_{w_2w_1} + 2a_{w_1w_2w_1} + 2a_{w_2w_1w_2w_1} + 3a_{w_1w_2w_1w_2w_1})$$

$$\times\ (1 + a_{w_2})a_{w_1}.$$

$$p_{\{w_1,w_2\}} = q_{\{w_1,w_2\}}.$$

The idempotents $\{q_J\}$ and $\{p_J\}$ for the 0-Hecke algebra of type (W,R), where $W = W(A_3)$ are as follows:

$$q_{\emptyset} = a_{w_1w_2w_1w_3w_2w_1}$$

$$q_{\{w_1\}} = -a_{w_2w_3w_2}(1 + a_{w_1})$$

$$q_{\{w_2\}} = (a_{w_1w_3} - a_{w_1w_3w_2w_1w_3})(1 + a_{w_2})$$

$$q_{\{w_3\}} = -a_{w_1w_2w_1}(1 + a_{w_3})$$

$$q_{\{w_1,w_2\}} = -a_{w_3}(1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1})$$

$$q_{\{w_1,w_3\}} = -(a_{w_2} + a_{w_2w_1w_3w_2})(1 + a_{w_3})(1 + a_{w_1})$$

$$q_{\{w_2,w_3\}} = -a_{w_1}(1 + a_{w_2})(1 + a_{w_3})(1 + a_{w_2})$$

$$q_{\{w_1,w_2,w_3\}} = \sum_{w \in W(A_3)} a_w.$$

$$p_{\emptyset} = a_{w_1w_2w_1w_3w_2w_1}$$

$$p_{\{w_1\}} = (a_{w_2w_3w_2} + a_{w_1w_2w_3w_2} + a_{w_2w_1w_2w_3w_2})(1 + a_{w_1})a_{w_2w_3w_2}.$$

$$p_{\{w_2\}} = (a_{w_1w_3} + a_{w_2w_1w_3} - 3a_{w_1w_3w_2w_1w_3})(1 + a_{w_2})a_{w_1w_3}.$$

$$p_{\{w_3\}} = (a_{w_1w_2w_1} + a_{w_3w_1w_2w_1} + a_{w_2w_3w_1w_2w_1})(1 + a_{w_3}) \times a_{w_1w_2w_1}.$$

$$p_{\{w_1,w_2\}} = (a_{w_3} + a_{w_2w_3} + a_{w_1w_2w_3})\left[1 + a_{w_1w_2w_1}\right]a_{w_3}.$$

$$p_{\{w_1,w_3\}} = (a_{w_2} + a_{w_1w_2} + a_{w_3w_2} + a_{w_1w_3w_2} + a_{w_2w_1w_3w_2}) \times(1 + a_{w_1})(1 + a_{w_3})a_{w_2}.$$

$$p_{\{w_2,w_3\}} = (a_{w_1} + a_{w_2w_1} + a_{w_3w_2w_1})\left[1 + a_{w_2w_3w_2}\right]a_{w_1}.$$

$$p_{\{w_1,w_2,w_3\}} = \sum_{w \in W(A_3)} a_w.$$

The idempotents $\{q_J\}$ for the 0-Hecke algebra of type (W,R) where $W = W(A_4)$ are as follows:

$$q_\emptyset = a_{w_1w_2w_3w_4w_1w_2w_3w_1w_2w_1}.$$

$$q_{\{w_1\}} = a_{w_2w_3w_2w_4w_3w_2}(1 + a_{w_1}).$$

$$q_{\{w_2\}} = (a_{w_1w_3w_4w_3} + a_{w_1w_4w_3w_2w_1w_3w_4w_3})(1 + a_{w_2}).$$

$$q_{\{w_3\}} = (a_{w_1w_2w_1w_4} + a_{w_1w_2w_3w_4w_1w_3w_2w_1})(1 + a_{w_3}).$$

$$q_{\{w_4\}} = a_{w_1w_2w_1w_3w_2w_1}(1 + a_{w_4}).$$

$$q_{\{w_1,w_2\}} = -a_{w_3w_4w_3}(1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1}).$$

$$q_{\{w_2,w_3\}} = (a_{w_1w_4} - a_{w_1w_2w_3w_4w_3w_2w_1})(1 + a_{w_2})(1 + a_{w_3}) \times (1 + a_{w_2})$$

$$q_{\{w_3,w_4\}} = -a_{w_1w_2w_1}(1 + a_{w_3})(1 + a_{w_4})(1 + a_{w_3}).$$

$$q_{\{w_1,w_3\}} = (a_{w_2w_4} - a_{w_2w_3w_4w_3w_2} - a_{w_2w_1w_3w_4w_3w_2} - a_{w_1w_2w_3w_4w_1w_3w_2})(1 + a_{w_1})(1 + a_{w_3}).$$

$$q_{\{w_2,w_4\}} = (a_{w_1w_3} - a_{w_1w_2w_3w_2w_1} - a_{w_1w_3w_2w_4w_3w_2} - a_{w_1w_3w_2w_4w_3w_2w_1})(1 + a_{w_2})(1 + a_{w_4}).$$

$$q_{\{w_1,w_4\}} = -(a_{w_2w_3w_2} + a_{w_2w_1w_3w_2w_4w_3w_1w_2})(1 + a_{w_1})(1 + a_{w_4}).$$

$$q_{\{w_1,w_2,w_3\}} = -a_{w_4}\left[1 + a_{w_1w_2w_3w_1w_2w_1}\right].$$

$$q_{\{w_2,w_3,w_4\}} = -a_{w_1}\left[1 + a_{w_2w_3w_2w_4w_3w_2}\right].$$

$$q_{\{w_1,w_2,w_4\}} = -(a_{w_3} + a_{w_3w_4w_2w_3})\left[1 + a_{w_1w_2w_1w_4}\right].$$

$$q_{\{w_1,w_3,w_4\}} = -(a_{w_2} + a_{w_2w_1w_3w_2})\left[1 + a_{w_1w_3w_4w_3}\right].$$

$$q_{\{w_1,w_2,w_3,w_4\}} = \sum_{w \in W(A_4)} a_w.$$

Appendix 4: THE CARTAN MATRIX OF THE 0-HECKE ALGEBRA.

Let $C = (c_{JL})_{J,L \subseteq R}$ be the Cartan matrix of the 0-Hecke algebra of type (W,R) over a field K. By (4.5.1),

$$c_{JL} = |Y_J \cap (Y_L)^{-1}| = c_{LJ}.$$

The Cartan matrix of the 0-Hecke algebra of type (W,R), where W is one of the finite Coxeter groups listed below are given in the next few pages.

(1) $W(A_1)$.

(2) $W(A_2)$.

(3) $W(A_3)$.

(4) $W(A_4)$.

(5) $W(B_2)$.

(6) $W(B_3)$.

(7) $W(G_2)$.

(8) $W(I_2(8))$.

(9) $W(A_1) \times W(A_1)$.

(10) $W(A_1) \times W(A_2)$.

(11) $W(A_2) \times W(A_2)$.

NOTATION: Let $R = \{w_1, w_2, \ldots w_n\}$, $|R| = n$. The rows and columns of C will be indexed by sets $(i_1, \ldots i_r)$, $1 \leq i_1 < i_2 < \ldots < i_r \leq n$, with $i_j \in Z$ for all j, where the element in the $(i_1, \ldots i_r) \times (j_1, \ldots j_s)$ position is the element c_{JL}, where $J = \{w_{i_1}, \ldots w_{i_r}\}$ and $L = \{w_{j_1}, \ldots w_{j_s}\}$.

(1) <u>$W(A_1)$</u>

$$\begin{array}{c|cc} & \emptyset & (1) \\ \hline \emptyset & 1 & 0 \\ (1) & 0 & 1 \end{array}$$

$$\begin{array}{c} \\ \emptyset \\ (1) \end{array} \begin{array}{c} \begin{array}{cc} \emptyset & (1) \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

(2) <u>$W(A_2)$</u>

$$\begin{array}{c} \\ \emptyset \\ (1) \\ (2) \\ (1,2) \end{array} \begin{array}{c} \begin{array}{cccc} \emptyset & (1) & (2) & (1,2) \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(3) <u>$W(A_3)$</u>

$$\begin{array}{c} \\ \emptyset \\ (1) \\ (2) \\ (3) \\ (1,2) \\ (1,3) \\ (2,3) \\ (1,2,3) \end{array} \begin{array}{c} \begin{array}{cccccccc} \emptyset & (1) & (2) & (3) & (1,2) & (1,3) & (2,3) & (1,2,3) \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(4) $W(A_4)$.

$$
\begin{array}{c}
\\
\emptyset \\ (1) \\ (2) \\ (3) \\ (4) \\ (1,2) \\ (2,3) \\ (3,4) \\ (1,3) \\ (2,4) \\ (1,4) \\ (1,2,3) \\ (1,2,4) \\ (1,3,4) \\ (2,3,4) \\ (1,2,3,4)
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc}
\emptyset & (1) & (2) & (3) & (4) & (1,2) & (2,3) & (3,4) & (1,3) & (2,4) & (1,4) & (1,2,3) & (1,2,4) & (1,3,4) & (2,3,4) & (1,2,3,4)
\end{array}
\\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 & 2 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 2 & 1 & 3 & 3 & 2 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 2 & 1 & 3 & 3 & 2 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

(5) $\underline{W(B_2)}$.

$$
\begin{array}{c|cccc|}
 & \emptyset & (1) & (2) & (1,2) \\
\hline
\emptyset & 1 & 0 & 0 & 0 \\
(1) & 0 & 2 & 1 & 0 \\
(2) & 0 & 1 & 2 & 0 \\
(1,2) & 0 & 0 & 0 & 1 \\
\hline
\end{array}
$$

(6) $\underline{W(B_3)}$.

$$
\begin{array}{c|cccccccc|}
 & \emptyset & (1) & (2) & (3) & (1,2) & (1,3) & (2,3) & (1,2,3) \\
\hline
\emptyset & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1) & 0 & 3 & 2 & 1 & 0 & 1 & 0 & 0 \\
(2) & 0 & 2 & 4 & 2 & 0 & 2 & 1 & 0 \\
(3) & 0 & 1 & 2 & 2 & 0 & 0 & 0 & 0 \\
(1,2) & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 0 \\
(1,3) & 0 & 1 & 2 & 0 & 2 & 4 & 2 & 0 \\
(2,3) & 0 & 0 & 1 & 0 & 1 & 2 & 3 & 0 \\
(1,2,3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}
$$

(7) $\underline{W(G_2)}$

$$
\begin{array}{c|cccc|}
 & \emptyset & (1) & (2) & (1,2) \\
\hline
\emptyset & 1 & 0 & 0 & 0 \\
(1) & 0 & 3 & 2 & 0 \\
(2) & 0 & 2 & 3 & 0 \\
(1,2) & 0 & 0 & 0 & 1 \\
\hline
\end{array}
$$

(8) $W(I_2(8))$

$$\begin{array}{c} \\ \emptyset \\ (1) \\ (2) \\ (1,2) \end{array} \begin{array}{c} \begin{array}{cccc} \emptyset & (1) & (2) & (1,2) \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 3 & 0 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(9) $W(A_1) \times W(A_1)$

$$\begin{array}{c} \\ \emptyset \\ (1) \\ (2) \\ (1,2) \end{array} \begin{array}{c} \begin{array}{cccc} \emptyset & (1) & (2) & (1,2) \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(10) $W(A_1) \times W(A_2)$.

$$\begin{array}{c} \\ \emptyset \\ (1) \\ (2) \\ (3) \\ (1,2) \\ (1,3) \\ (2,3) \\ (1,2,3) \end{array} \begin{array}{c} \begin{array}{cccccccc} \emptyset & (1) & (2) & (3) & (1,2) & (1,3) & (2,3) & (1,2,3) \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(11) $\underline{W(A_2)} \times \underline{W(A_2)}$.

$$
\begin{array}{c|cccccccccccccccc|}
 & \emptyset & (1) & (2) & (3) & (4) & (1,2) & (1,3) & (1,4) & (2,3) & (2,4) & (3,4) & (1,2,3) & (1,2,4) & (1,3,4) & (2,3,4) & (1,2,3,4) \\
\hline
\emptyset & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1) & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(2) & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(3) & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(4) & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1,2) & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1,3) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1,4) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
(2,3) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
(2,4) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
(3,4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
(1,2,3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
(1,2,4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
(1,3,4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
(2,3,4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
(1,2,3,4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

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